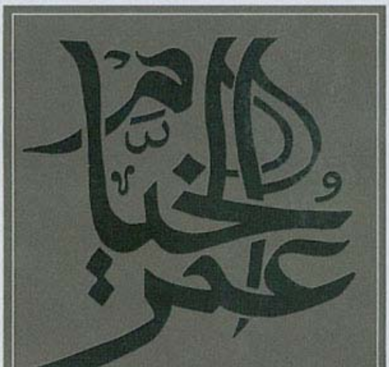
The Great Books of Islamic Civilization

An Essay by the Uniquely Wise 'Abel Fath Omar Bin Al-Khayyam on Algebra and Equations

Omar Al-Khayyam

Algebra wa Al-Muqabala

Translated by Professor Roshdi Khalil
Reviewed by Professor Waleed Deeb



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The Center for Muslim Contribution to Civilization

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In the Name of God, the Beneficent, the Merciful

Foreword

THE interrelationship and interaction of human cultures and civilizations has made the contributions of each the common heritage of men in all ages and all places. Early Muslim scholars were able to communicate with their Western counterparts through contacts made during the Crusades; at Muslim universities and centres of learning in Muslim Spain (al-Andalus, or Andalusia) and Sicily to which many European students went for education; and at the universities and centres of learning in Europe itself (such as Salerno, Padua, Montpellier, Paris, and Oxford), where Islamic works were taught in Latin translations. Among the Muslim scholars well-known in the centres of learning throughout the world were al-Rāzī (Rhazes), Ibn Sīnā (Avicenna), Ibn Rushd (Averroes), al-Khwārizmī and Ibn Khaldūn. Muslim scholars such as these and others produced original works in many fields. Many of them possessed encyclopaedic knowledge and distinguished themselves in many disparate fields of knowledge.

The Center for Muslim Contribution to Civilization was established in order to acquaint non-Muslims with the contributions Islam has made to human civilization as a whole. The Great Books of Islamic Civilization Project attempts to cover the first 800 years of Islam, or what may be called Islam's Classical Period. This project aims at making available in English and other European languages a wide selection of works representative of Islamic civilization in all its diversity. It is made up of translations of original Arabic works that were produced in the formative centuries of Islam, and is meant to serve the needs of a potentially large readership. Not only the specialist and scholar, but the non-specialist with an interest in Islam and its cultural heritage will be able to benefit from the series. Together, the works should serve as a rich source for the study of the early periods of Islamic thought.

In selecting the books for the series, the Center took into account all major areas of Islamic intellectual pursuit that could be represented. Thus the series includes works not only on better-known subjects such as law, theology, jurisprudence, history and politics, but also on subjects such as literature, medicine, astronomy, optics and geography. The specific criteria used to select individual books were these: that a book should give a faithful and comprehensive account of its field; and that it should be an authoritative source. The reader thus has at his disposal virtually a whole library of informative and enlightening works.

Each book in the series has been translated by a qualified scholar and reviewed by another expert. While the style of one translation will naturally differ from another as do the styles of the authors, the translators have endeavoured, to the extent it was possible, to make the works accessible to the common reader. As a rule, the use of footnotes has been kept to a minimum, though a more extensive use of them was necessitated in some cases.

This series is presented in the hope that it will contribute to a greater understanding in the West of the cultural and intellectual heritage of Islam and will therefore provide an important means towards greater understanding of today's world.

May God Help Us!

Muhammad bin Hamad Al-Thani
Chairman of the Board of Trustees

About this Series

This series of Arabic works, made available in English translation, represents an outstanding selection of important Islamic studies in a variety of fields of knowledge. The works selected for inclusion in this series meet specific criteria. They are recognized by Muslim scholars as being early and important in their fields, as works whose importance is broadly recognized by international scholars, and as having had a genuinely significant impact on the development of human culture.

Readers will therefore see that this series includes a variety of works in the purely Islamic sciences, such as Qurʾān, *ḥadīth*, theology, prophetic traditions (*sunna*), and jurisprudence (*fiqh*). Also represented will be books by Muslim scientists on medicine, astronomy, geography, physics, chemistry, horticulture, and other fields.

The work of translating these texts has been entrusted to a group of professors in the Islamic and Western worlds who are recognized authorities in their fields. It has been deemed appropriate, in order to ensure accuracy and fluency, that two persons, one with Arabic as his mother tongue and another with English as his mother tongue, should participate together in the translation and revision of each text.

This series is distinguished from other similar intercultural projects by its distinctive objectives and methodology. These works will fill a genuine gap in the library of human thought. They will prove extremely useful to all those with an interest in Islamic culture, its interaction with Western thought, and its impact on culture throughout the world. They will, it is hoped, fulfil an important rôle in enhancing world understanding at a time when there is such evident and urgent need for the development of peaceful coexistence.

This series is published by the Center for Muslim Contribution to Civilization, which serves as a research centre under the patronage of H.H. Sheikh Muhammad bin Hamad al-Thani, the former Minister of Education of Qatar who also chairs the Board of Trustees. The Board is comprised of a group of prominent scholars. These include His Eminence Sheikh Al-Azhar, Arab Republic of Egypt, and Dr Yousef al-Qaradhawi, Director of the Sira and Sunna Research Center. At its inception the Center was directed by the late Dr Muhammad Ibrahim Kazim, former Rector of Qatar University, who established its initial objectives.

The Center was until recently directed by Dr Kamal Naji, the Foreign Cultural Relations Advisor of the Ministry of Education of Qatar. He was assisted by a Board comprising a number of academicians of Qatar University, in addition to a consultative committee chaired by Dr Ezzeddin Ibrahim, former Rector of the University of the United Arab Emirates. A further committee acting on behalf of the Center has been the prominent university professors who act under the chairmanship of Dr Raji Rammuny, Professor of Arabic at the University of Michigan. This committee is charged with making known, in Europe and in America, the books selected for translation, and in selecting and enlisting properly qualified university professors, orientalists and students of Islamic studies to undertake the work of translation and revision, as well as overseeing the publication process.

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Translator's Introduction

When I accepted the challenge of translating Omar Al-Khayyam's book from Arabic to English, I told myself this would be an easy job to do. After all, I consider myself to be good at both mathematics and the Arabic language. The shock came when I discovered that first I had to translate from the Arabic of Omar Al-Khayyam to my Arabic. That was the difficult part of my job. Another shock was to discover my ignorance of the scientific achievements of great scientists such as Omar Al-Khayyam. Going through the book, reading theorems, proofs and remarks, I realized that I was reading the work of a great mathematician. The preciseness of the statements and the accuracy of the proofs made me think that I was reading an article in a high-ranking recent mathematical journal.

Omar Al-Khayyam was born in the middle of the eleventh century in the city of Nishapur. He was a poet and a mathematician. I read some of his poetry when I was an undergraduate student. At that time I did not know that he was a mathematician and had written books on mathematics.

Omar Al-Khayyam's book mainly deals with equations of degree at most three:

$$ax^3 + bx^2 + cx + d = 0$$

including all cases where some of the integer coefficients a, b, c, d equal zero.

At the time of Omar Al-Khayyam, the two equations

$$ax^3 + bx^2 + cx + d = 0$$
$$ax^3 + bx^2 + cx = d$$

were regarded as two different cases of equations of degree three.

Al-Khayyam's book studies and presents the following:

1. Third degree equations that can be reduced to equations of degree two.
2. Third degree equations that consist of three terms.
3. Third degree equations that consist of four terms.
4. Equations that involve the reciprocal of the unknown (variable).
5. The problem of dividing a quarter of a circle into two parts with a given ratio.
6. A discussion of some results of Abu-Aljood Ben Al-Laith.

Omar Al-Khayyam used geometry, especially conic sections, to prove his results.

I learned so much from the project of translating Omar Al-Khayyam's book – *Algebra wa Al-Muqabala* (Algebra and equations). Just after I finished translating the book, I met Professor Roshdi Rashid, in Amman at a conference on the history

of Arab sciences. He drew my attention to the fact that there are actually two different manuscripts of Al-Khayyam's book. One copy is in Aleppo, the one that I translated into English. The second copy is in France, the one he translated into English. I thank Professor Rashid very much for his comments and the valuable information he supplied me with.

I wish to thank Professor Raji Rammuny who suggested my name for the project of translating Al-Khayyam's book, thus offering me the chance to explore part of the work of a great scholar of my culture.

I thank all those who helped me in my task in one way or another, in particular my students S. Awamlah and A. Khawalda, who helped me in drawing the graphs for the book.

Algebra wa Al-Muqabala

An Essay by the Uniquely Wise 'Abel Fath Omar Bin Al-Khayyam on Algebra and Equations

One of the educational notions needed in the branch of philosophy known as mathematics is the art of algebra and equations, invented to determine unknown numbers and areas. It involves problems that reflect difficult propositions; most people studying such problems have been unable to solve them. In the case of the ancient ones [researchers], none of their work has reached us, either because they did not solve these problems despite trying, or because they did not need to solve them, or simply because their work was lost.

As for the modern ones, Mahani tried to analyze algebraically the proposition used by Archimedes as a postulate in proposition four of the second article of his algebra book on spheres and cylinders. In his analysis, he discovered equations involving squares and cubes of numbers that he could not solve, though he thought deeply about them. So, he concluded that such equations are impossible to solve. No one could solve such equations, until the genius "Abu Jafar Al-Khazin" solved them using conic sections.

Later on, a group of geometers needed certain classes of these equations. Some of them (the geometers) solved certain types of these problems. But none of them did any sound work concerning the classification and sub-classification of such problems or the proof of any, except for two classes that I will mention later in this book. And I was, and remain, very keen to classify the problems and indicate (by proof) those that can be solved and those that are impossible to solve, since I knew the very need of it in solving other (open) problems.

I was unable to devote myself completely to achieving this worthy task, or to pursuing my ideas generally, because the demands of my daily life were a necessary diversion.

We have been afflicted in our time by a lack of scientists, except for a particular group, few in number but many in the troubles that beset them, whose concern is to exploit any gleam of trouble-free time to achieve and articulate some branches of science.

Many of those who pretend to be wise men in our time defraud the truth with falsity and do not seek to move forward the frontiers of knowledge, preferring instead to use the little they know of science for low materialistic goals. And once they meet someone who sincerely wishes to acquire facts and who prefers truth, trying to reject falsity and avoid deception, they make fun of him and ridicule him. So may God be our helper and our comforter.

God gave me the opportunity of being with our unique great master, the head judge, the scholar Imam Abi Tahir, may God keep his high position, and keep silent his enviers and enemies. Once I despaired to meet someone like him, perfect in every virtue, theoretical or practical, who can work deeply in science, verify others' work and seeks the welfare of everyone of his kind; I was so delighted to see him. I achieved fame through his companionship. My affairs were glorified by his illuminations, and my support intensified through his grace. It was my opportunity to benefit from my new status.

So I started to summarize what I can verify of deep knowledge so as to be closer to the master (Abi Tahir). Since the priority is mathematics, I started to list the algebraic propositions.

I adhered to the guidance of God, and I entreated God to grant me success in verifying my scientific research and the important research of those before me, grasping the most trustworthy of God's protection. For He is the one who answers our prayers and on whom we depend.

With the help of God, and with his gentle guidance, I say:

The study of algebra and equations is a scientific art. The ingredients are the absolute numbers and unknown measurable quantities, which are related to a known quantity. Each known thing is either a quantity or a unique relation that can be determined by careful examination.

By quantities we mean continuous quantities, and they are of four types: line, surface, solid, and time – as mentioned briefly in *Categories*, the book of Aristotle, and in detail in his other book, *Metaphysics*. Some (researchers) consider place to be a continuous quantity of the same type as surface. This is not the case, as one can verify. The truth is: space is a surface with conditions, whose verification is not part of our goal in this book. It is not usual to mention time as an object in algebraic problems. But if it were mentioned, it would be quite acceptable.

It is a practice of the algebraists (in their work) to call the unknown to be determined an object (variable), and the product of the object by itself a square (maal). The product of the object by its square is called a cube, and the product of the square by the square: square-square (maal-maal); the product of cube by square: square-cube, and the product of cube by cube: cube-cube, and so on.

It is known from the *Elements*, the book of Euclid, that these ranked products are all proportional in the sense that the ratio of one to the root is as the ratio of root to square, is as the ratio of square to cube. So the ratio of the numbers to roots is as the ratio of roots to squares, is as the ratio of squares to cubes, is as the ratio of cubes to square-squares, and so on.

It has to be clear that for anyone to be able to understand this essay, he has to be acquainted with the two books of Euclid (the *Elements* and the *Data*) and two chapters of the book of Apollonius on *Cones*. Whoever lacks knowledge of any of these three references, will not understand this essay. I have taken pains in trying not to refer to any article or book except those three books.

Algebraic solutions are achieved using equations. I mean, as is well known, by equating the ranks (powers) one with the other. If an algebraist uses square-square in areas, then this would be figuration not fact, since it is impossible for square-square to exist in measurable quantities. What we get in measurable quantities is first one dimension, which is the “root” or the “side” in relation to its square; then two dimensions, which represent the surface and the (algebraic) square representing the squared surface (rectangle); and finally three dimensions, which represent the solid. The cube in quantities is the solid bounded by six squares (parallelepiped), and since there is no other dimension, the square of the square does not fall under measurable quantities let alone higher powers.

If it is said that the square of the square is among measurable quantities, this is said with reference to its reciprocal value in problems of measurement and not because it in itself is measurable. There is a difference between the two cases.

The square of the square is, therefore, neither essentially nor accidentally a measurable quantity and is not as even and odd numbers, which are accidentally included in measurable quantities, depending on the way in which they represent continuous measurable quantities as discontinuous.

Of the four (geometrical) equations that involve the absolute numbers, sides, squares and cubes, the books of the algebraists contain only three of these equations involving numbers, sides and squares. But we will give the methods by which one can determine the unknown using equations that involve the four measurable quantities that we mentioned, I mean: the number, the object, the square and the cube. Whatever can be proved using the properties of the circle, I mean using the two books of Euclid, the *Elements* and the *Data*, will be given simpler proofs. But those that cannot be proved except by using conic sections will be proved using two articles on conics.

As for the proofs of these types (of problems): if the problem is concerned only with the absolute number, then we cannot (in general) supply the proofs (and no one who works in this industry can). Hopefully, someone who comes after us will be able to (supply the proofs). As for the first three ranks: number, object and square, we will supply the proofs, and I may refer to numerical proofs of problems that can be proved using the book of Euclid (the *Elements*).

You have to know that the geometrical proofs of these problems do not dispense the numerical proofs if the topic is the number not the measurable quantities. Can you not see that Euclid proved certain equations to find the required rational measurable quantities in chapter five of his book (the *Elements*), and then resumed his proof of such problems, in chapter seven of his book, to determine these required ratios, if the topic is some number.

Equations involving these four types are either simple or multi-term equations. The simple equations are of six types:

(1) Number equals root
(2) Number equals square
(3) Number equals cube
(4) Roots equal square
(5) Squares equal cube
(6) Roots equal cube.

Three of these six equations are mentioned in the books of the algebraists. They (the algebraists) said: the ratio of the object to the square is as the ratio of the square to the cube. So equating the square to the cube is as equating the object to the square. Further, the ratio of the number to the square is as the ratio of the root to the cube. So it follows that the equation of the number and the square is as the equation of the root and the cube. They (the algebraists) did not prove that using geometry.

As for the number that equals the (volume) cube, there is no way to determine it (the number) except through mathematical induction. If a geometrical method is to be used to determine the number, then it can only be done through conic sections.

As for multi-term equations, they are of two classes: three-term equations and four-term equations. The three-term equations are of twelve types. The first three of them are:

(1) Square and a root equal a number
(2) Square and a number equal root
(3) Root and a number equal square.

These three have been mentioned in the algebraists' books, including their proofs using geometry, not using numbers. The second three (of the three-term equations) are:

(1) Cube and square equal root
(2) Cube and root equal square
(3) Cube equals root and square.

The algebraists said that these three equations are equivalent to the first three, each to the corresponding one. I mean: cube and root equal square is equivalent to square and number equal root. And the other two are the same. They (the algebraists) did not prove them if the topic (unknown) of the problem is area. But they did solve them if the unknown is number, as is clear in the book of *Elements*. I will prove the geometrical ones.

The other six types of the twelve types are:

(1) Cube and root equal number
(2) Cube and number equal root
(3) Number and root equal cube
(4) Cube and square equal number
(5) Cube and number equal square
(6) Number and square equal cube.

These six types were never mentioned in their (the algebraists') books, except for one, where the proof was not complete. I will prove all of these types using geometry, not numeric. The proofs of these six types can only be deduced through the properties of conic sections.

As for the four-term equations, they consist of two groups. The first group, where three terms equal one term, contains four types:

(1) Cube, square and root equal number
(2) Cube, square and number equal root
(3) Cube, root and number equal square
(4) Cube equals root, square and number.

The second group, in which two terms equal two terms, is of three types:

(1) Cube and square equal root and number
(2) Cube and root equal square and number
(3) Cube and number equal root and square.

These are the seven types of the four-term equations, none of which can be solved except through geometry.

One of those (algebraists) who lived before us needed one type of one part of these equations, as I will mention. The proof of these types cannot be produced except through conic sections. We will prove the twenty-five types of these equations, one by one, asking help from God, for those who sincerely depend on Him will get help and guidance.

First type of simple equations: Root equals a number ($ax = b$)

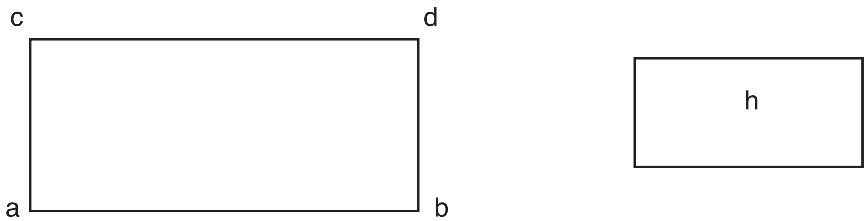
The root is necessarily known; this applies to numbers and areas.

Second type: Number equals a square ($x^2 = b$)

So the square is known, being equal to the known number. There is no way to find the root except by trial. Those who know that the root of twenty-five is five know that by induction, not through a deduced formula, and one need pay no attention to people who differ in this matter. People of India have ways of finding the side of a square (knowing the area) and of a cube (knowing the volume) and these methods are based on simple induction, that depends on knowing the squares of the nine (numbers). I mean the square of one, two, three, and their products (that is to say the product of two by three and so on). We have written an article in which we prove the validity of these methods and show how it leads to the required results. We enriched its types, I mean by finding the sides of square-square, square-cube, and cube-cube, and so on. We were the first to do that. These proofs are numerical ones and are based on the numerical part of the book of *Elements*.

The following is the proof of the second type using geometry:

Draw the line ab whose length equals the given number and a line ac (of unit length) perpendicular to ab and then complete the surface (rectangle) ad. The area of the surface ad is equal to the given number. We construct a square whose area equals the area of ad; call it h, as shown by Euclid, proposition zd, article b of his book. The (area of) square h equals the given number, which is known. Its side is also known. Examine the proof given by Euclid. This is the required result.

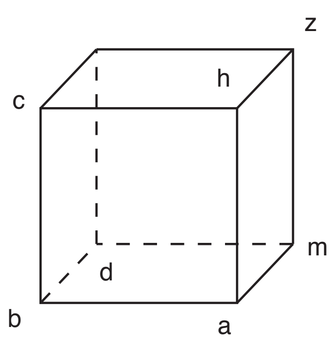


Whenever we say (in this essay) a number equals a surface (area of a rectangle), we mean there is a right-angle surface (rectangle) with one side equal to one unit and the other equals the given number, and each unit part of the area equals the other side, which we assumed to be one.

Third type: A number equals a cube

If the number is known, the cube (volume) is known and the only way to find its side is by induction. The same applies to higher orders like square-square, square-cube, and cube-cube, as we mentioned earlier.

But, using geometry we construct the square ad to be of unit side, I mean ac and bc both equal one. Then, we draw a perpendicular to the surface ad at the point b, call it bc, of length equal to the given number, as Euclid showed in section eleven of his book. We complete the parallelepiped abchzm.



It is known then that the area of this surface equals the given number. Now, we construct a cube of the same volume as this parallelepiped. Such a construction cannot be done without using properties of conic sections. We delay this until we introduce the background for this study. When we mentioned that a number equals a parallelepiped we mean there is a parallelepiped (with right angles) with a unit square base and its height equals the given number.

Fourth type: Square equals five of its root ($x^2 = 5x$)

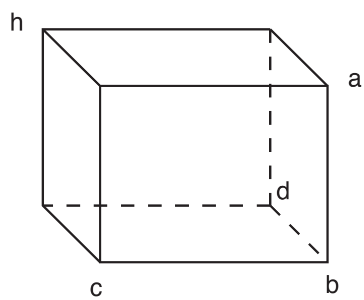
So the number of the roots (five) equals the root of the square. The numerical proof is as follows: If the root is multiplied by itself you get the square. But if this root is multiplied by five you get the square. So it must be five.

The geometric proof is similar to the numerical one. Just construct a square whose area is equal to five times the length of its side.

Fifth type: Roots equal cube ($ax = x^3$)

Numerically: clearly, this is equivalent to a number equals a square. As an example: four times a root equals a cube, which is equivalent to four equals a square.

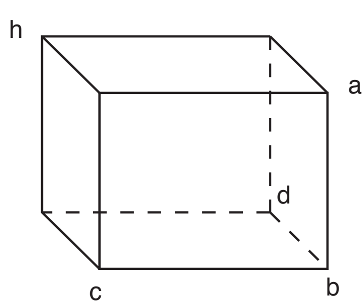
But geometrically: we construct a cube abcdh whose volume equals four times the length of its side ab. So multiplying the length of the side ab by four gives the volume of the cube. But, multiplying the length of ab by the area of the square ac gives the volume of the cube again. So, the area of the square of ac equals four.



Sixth type: Squares equal a cube ($ax^2 = x^3$)

This is equivalent to a number equals a root ($a = x$). To prove this numerically: the ratio of a number to a root is the same as the ratio of a square to the cube, as shown in chapter eight of Euclid's book, the *Elements*.

Geometrically: we construct a cube abcdh whose volume is equal to the number of the squares of its side; for example twice the square of its side.



If we multiply the area of ac by two, we get the volume of the cube abcdh. Also, if we multiply the area of ac by (the length of) bd we get the volume of the cube again. So (the length of) bd must be equal to two. This is what we wanted to show.

Whenever we say in this article 'the squares of the cube', we mean the squares of its sides.

Now, we have finished the simple equations. Let us discuss the first three of the twelve types of three-term equations.

First type: Square plus ten times its root equals thirty-nine ($x^2 + 10x = 39$)

Multiply half the number of the roots by itself. Add the result to the number of the roots. Then, subtract from the root of the outcome half the number of the roots. What is left is the root of the square.

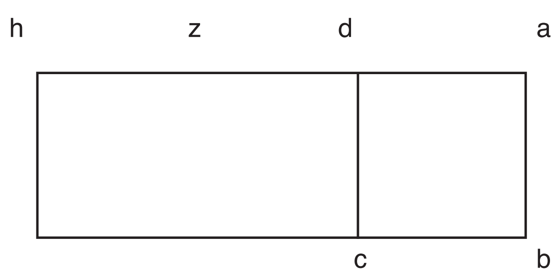
The number must satisfy two conditions. The first is: the number of the roots must be even in order to have a half. The second is: the square of half the number of the roots plus the given number is a complete square. Otherwise, it is impossible to solve the problem numerically.

However, using geometry, none of the cases is impossible to solve. The numerical solution is much simpler if one visualizes the geometric solution.

The geometric solution is as follows:

We let the area of the square ac plus ten times its root equal thirty-nine. But ten times the root equals the area of the rectangle ch, so the side dh equals ten.

We bisect dh at z. Since dh was divided into two equal parts at z, and extended to include ad, then the product of ha and ad (this product equals the area of bh) plus the square of dz, equals the square of za.

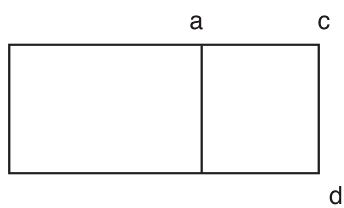


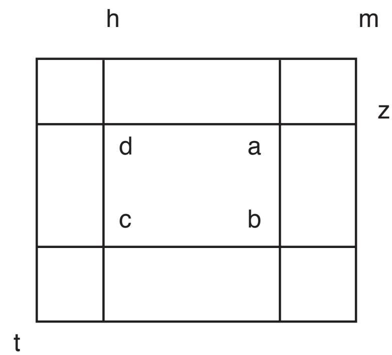
But, the square of dz, which equals half the number of the roots, is known. And the area of bh, which is the given number, is also known. So the square of za is known, and so is the line za. If we subtract zd from za we are left with ad, and so ad is known.

There is another proof:

We construct the square abcd, and extend ba to m so that ma equals one fourth of the number of the roots, which is two and a half. We extend da to z so that za equals one fourth of the number of the roots. We construct similar lines from all corners of the square abcd.

We complete the rectangle mt, which is a square because zm is a square, ac is a square and ct is a square as shown in chapter six of the book of *Elements*.

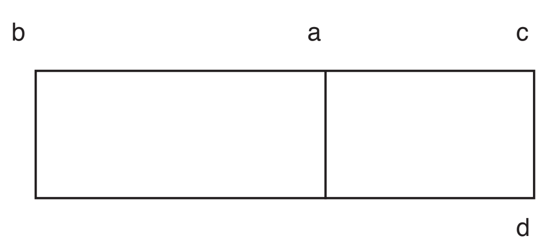




Each of the four squares at the corners of the big square (abcd) has area equal to the square of two and a half. So the sum of all (areas) squares is twenty-five, which equals the square of half the number of the roots. Further, the (area of) rectangle zb equals two and a half of the roots (sides) of the square of ac, because za equals two and a half, so the (area of the) four rectangles equals ten roots (side) of the square of ac. But the square of ac and ten of its roots was assumed to equal thirty-nine. Hence the square of mt is sixty-four. Take its root and subtract five. What is left is ab.

Now, if the (length) line ab is assumed to equal ten, and we need to have a square for which the product of the length of its side by the length of ab equals the given number, then we let the given number equal the area of a rectangle h.

We construct, on the line ab, a rectangle equal (in area) to the rectangle h, then we construct a square on the side of the new rectangle, say bd, as shown by Euclid in chapter six of his book, the *Elements*. So the added square ad and its side ac are known, as shown in *Data*.



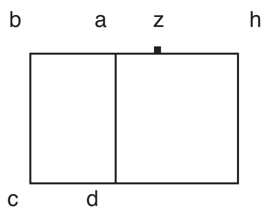
Second type: Square plus a number equals roots ($x^2 + a = bx$)

In this case, the number should not exceed the square of half the number of the roots; otherwise the problem has no solution.

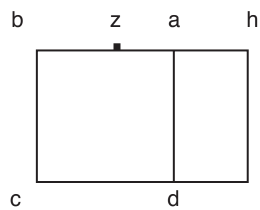
In case the number is equal to the square of half the number of the roots, then half the number of the roots is the root of the square. In case it is less than the square of half the number of the roots, then subtract the number from the square of half the number of the roots, and take the root of the result. Add (or subtract from) the outcome to half the number of the roots. What we get after adding or subtracting is the root of the square.

The numerical proof can be visualized, once the geometric proof is understood:

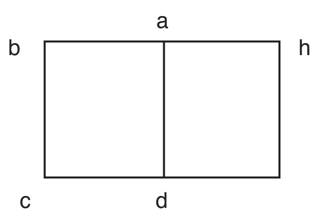
We construct the square abcd, and the rectangle hd on the side ad so that, for example, the area of hc is ten times the side of the square ac. This implies hb equals ten. In the first case, ab equals half bh, and in the second case, greater than its half, and in the third, less than its half. In the first case, ab will be equal to five; while in the second and the third cases, we bisect hb at z.



First



Second



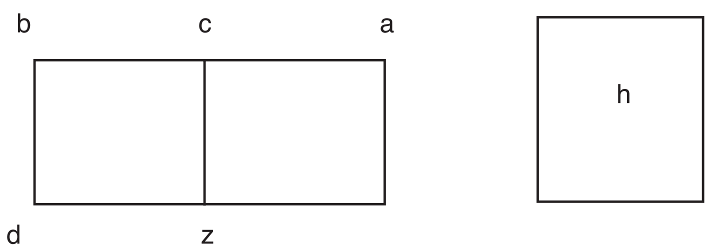
Third

So, the line hb is divided into two halves at z and into unequal parts at a. So the product of ha and ab added to the square of za equals the square of zb, as shown in the second chapter of the book of *Elements*.

But the product of ha and ab is the given number, which is known. If this product is subtracted from the square of zb, which equals half the number of the roots, we are left with the square of za, which is known. In the third case, subtract the product from zb. In the second case add to the product az. The outcome will be ab, which is the required result. If you wish, you can prove it in other ways, but we do not want to present them to avoid lengthy treatment.

But if it is assumed that the length of ab is ten, say, and a segment is to be cut out from it (ab), so that the product of ab times (the length of) that line equals the square of that line plus the area of another rectangle, not larger than the square of half ab, I mean the given number which is the area of the rectangle h, and we want to cut from ab a line whose square plus the rectangle h, equals the product of ab times (the length of) that line, we add to the given line ab a rectangle, say az, whose area equals the known area of h minus the area of a square, say cd, which is possible because the area of h is not larger than the square of half ab, as shown by Euclid in chapter six of his book (the *Elements*). So the side of cb is known from the given data, and that is the required result.

It seems that this type has many kinds, some of them impossible to solve. You can find the conditions on the number, which make the problem solvable, based on what we have shown in our discussion of the first type.



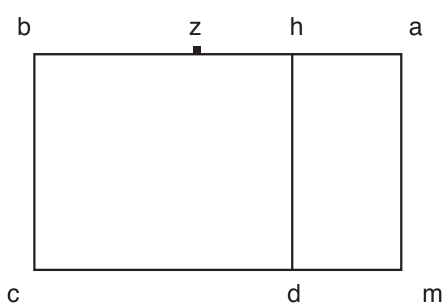
Third type: Number and a root equal a square ($a + bx = x^2$)

Add the square of half the number of the roots to the number. Then take the root of this sum and add it to half the number of the roots. The result will be the root of the square.

Proof: The square abcm equals five times its root plus six ($x^2 = 5x + 6$). We subtract the number, which is the rectangle ad. We are left with the rectangle hc, which equals the number of the roots, which is five.

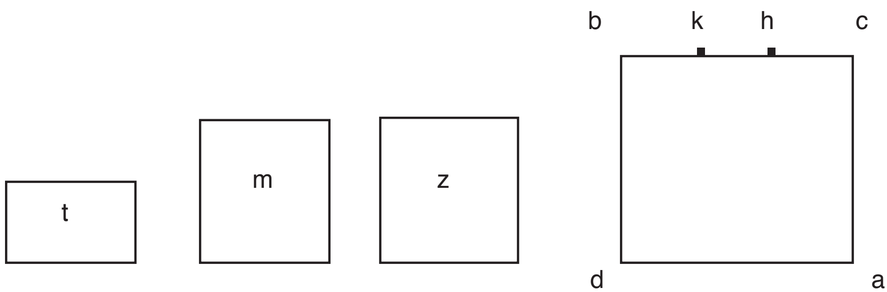
So the length of hb equals five. We divide hb into two equal parts at z. Hence, the line ab is divided in two equal parts at z. Add to it ha by extension.

So the product of ba and ah (which is equal to the area of the known rectangle ad) plus the known square of hz, equals the square of za. Consequently, since the square of za, za itself, and zb are all known, we conclude that ab is known. There are other methods to prove this. We accept this one.



But if it is assumed that hb equals the number of the roots, and it is required to find a square for which the square plus its side equals the number of the sides plus the given number, then abcd is the required square.

To see that, assume that the given number is the area of the rectangle t, and the square that is equal to it is m. We construct an equal square (to m), say z. We construct kc to equal the side of z; then we complete the square abcd. The square abcd is the required square.



We have shown that in this third type, no case is impossible to solve. This is true also for the first type. For the second type there are cases with no solution, and it involves cases that do not occur in the first and the third type.

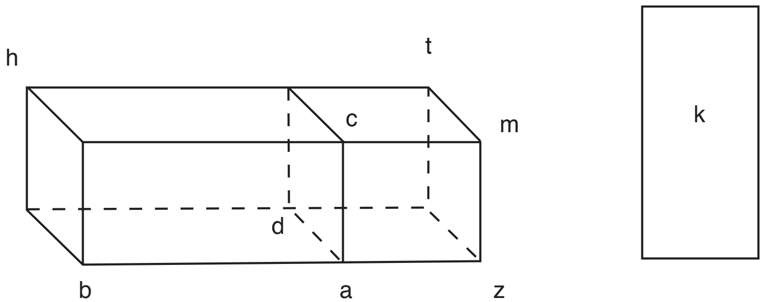
Equations of the third degree that can be reduced to second degree equations

The proof that the second three equations are equivalent to the first three is as follows.

First type: Cube and squares equal roots ($x^3 + ax^2 = bx$)

We construct a cube abcdh. Extend ab to z so that az equals the given number of the squares. Then complete the parallelepiped azmtcd beside the cube ah, as usual. The parallelepiped at equals the number of the squares. So the parallelepiped bt, which equals the cube plus the number of the given squares, equals the number of the given roots. Construct the surface (rectangle) k to equal the number of the assumed roots. The root is the side of the cube, and it is ad. Hence, if the area of the rectangle k is multiplied by the length of ad, the result will be equal to the number of the given roots. Now, if the (area) rectangle mb is multiplied by ad, we get the cube plus the assumed number of squares. But these two quantities are equal.

I mean, the parallelepiped bt and the parallelepiped constructed on the base k with height ad have the same volume. Since they have equal heights, they must have equal bases. The base of mb is the square cb with the square ma, which is the number of the roots of the assumed squares.

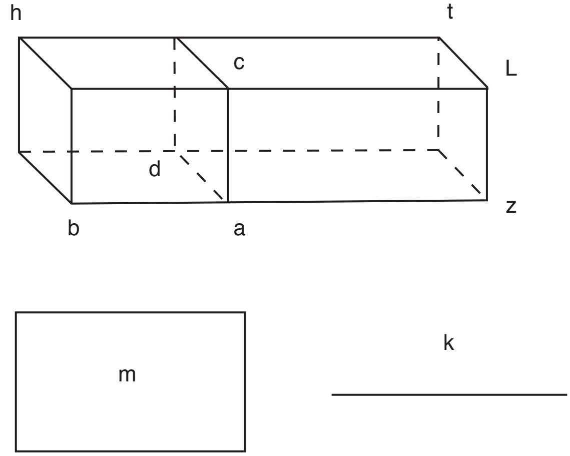


Hence, k, which is the assumed number of the roots, equals a square plus the assumed number of the roots of the squares. That is what we wanted to show.

Example: A cube plus three squares equals ten roots is equivalent to a square plus three roots equals ten.

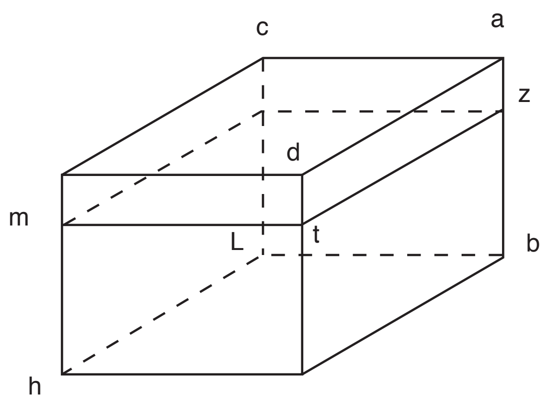
Second type: Cube and two roots equals three squares, which is equivalent to square plus two equals three roots ($x^3 + 2x = 3x^2 \leftrightarrow x^2 + 2 = 3x$)

Proof: We construct the cube abcdh so that the cube plus two of its roots equals three squares. We then construct a square m to equal the square cb, and we set k equal to three. So the product of m with k equals three squares of the root of the cube ah. We construct on ac a rectangle that equals two, then we complete the solid (parallelepiped) azctd. Then azctd equals the number of the roots. But if the (length) line bz is multiplied by the square ac we get the (volume) parallelepiped bt. However, the parallelepiped at equals the number of roots. So the parallelepiped bt equals the cube plus the number of its sides. So the parallelepiped bt equals the number of the squares, and hence the length of bz is equal to three, as was shown earlier. The rectangle bl is a square plus two, so a square plus two equals three roots because the rectangle bl is the product of ab by three and that is what we wanted to show.



Third type: Cube equals a square and three roots ($x^3 = x^2 + 3x$) which is equivalent to square equals a root and three ($x^2 = x + 3$)

Proof: Construct the cube abcdh to equal a square and three of its sides. Then we remove from ab (which is the side of the cube) the segment az, whose length equals the number of the squares which is equal to one. Complete the parallelepiped aztmc so its volume equals the assumed number of the squares. What remains is the parallelepiped zh, which is equal to the number of the assumed sides. The ratio of the (volumes) of the two parallelepipeds equals the ratio of the base of zc to the base of zl, as was shown in chapter eleven of the book of *Elements*, since their heights are equal. But the (area) of the rectangle zc is one root of the square cb, and (area of) zl equals the number of the roots which is three. Thus, the square of cb equals a root and three, and that is what we wanted to show.



If our explanations of the proofs are not understood, then the proofs may appear not to be correct, though they were difficult to explain.

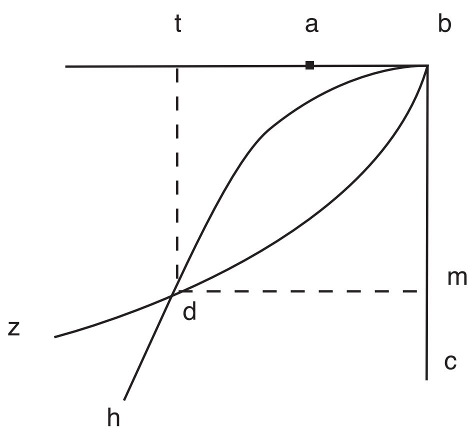
Equations of degree three, composed of three terms

After introducing these types, which we were able to prove using the properties of circles, mentioned in the book of Euclid, let us move now to types whose proofs can only be given using the properties of conic sections. There are fourteen of these types. One is a single-term equation, which is a cube equals a number, six consist of three-term equations and seven consist of four-term equations. Let us give an introduction based on the book of conic sections to serve as background for the student so that we will not need to refer to more than the three mentioned books, that is the two books of Euclid, the *Elements* and the *Data*, and two articles of the book of conic sections.

Introduction 1 (Lemma 1): We need to find two lines between two given lines so that the four are proportional.

Let the two given lines be ab, bc, making a right angle at b. We construct the parabola bdh, with vertex at b, axis bc and its right side is bc. So the conic section bdh is known because its vertex and axis are known, and the magnitude of its right side is known. Further, the conic will be tangent to the line ba because b is a right angle and it is equal to the angle of order, as shown in article a of the book of conics.

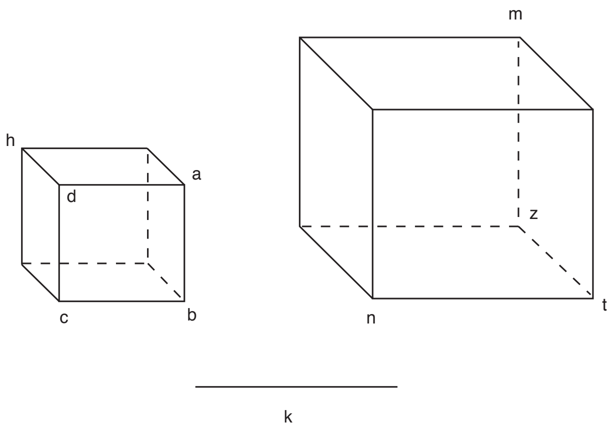
Similarly, we construct another parabola, bdz, with vertex at b, axis ab, and its orthogonal side is ab. The conic bdz is tangent to bc as was shown by Apollonius in proposition no of article a. The two sections must intersect at the point d, say. The location of d is known because the location of the two sections is known. We draw from d two lines dm, dt orthogonal to ab, bc respectively. So their magnitudes are known, as shown in the *Data*. I claim: the four lines ab, bm, bt, bc are proportional.



Proof: The square of md equals the product of bm and bc, because the line dm is one of the lines of order of the section bdh. It follows that the ratio of bc to md (which is equal to the line bt) is as the ratio of bt to mb. And the line dt is one of the lines of order of the section bdz, so the square of dt (which is equal to bm) equals the product of ba and bt. Thus, the ratio of bt to bm is as the ratio of bm to ba. Hence the four lines are successive and proportional, and the magnitude of the line dm is known because it is drawn from a given point to a given line with a given angle. Further, the magnitude of dt is known. So the two lines bm and bt are of known magnitude and they are middle in ratio between ab and bc; I mean the ratio of ab to bm is the same as the ratio of bm to bt and is as the ratio of bt to bc, and that is what we wanted to show.

Introduction 2 (Lemma 2): Given a square abcd, which is a base of a solid (parallelepiped) abcdh with parallel surfaces and right angles; and given a square nm. We need to construct a parallelepiped such that its base is nm and its volume is equal to the volume of abcdh.

We make the ratio of ab to nz equal to the ratio of mz to k, and the ratio of ab to k equal to the ratio of zt to hd. Then make zt perpendicular to the square nm at z, and complete the parallelepiped nztm. I claim that the volume of this parallelepiped equals the volume of the given one.



Proof: The ratio of the (area of the) square ac to the (area of the) square nm is the same as ab to k. Thus, the ratio of the (area of the) square ab to the (area of the) square nm is the same as the ratio of zt (which is the height of the parallelepiped ntm) to dh (which is the height of the parallelepiped bh). Hence, the two parallelepipeds have the same volume because their bases are equivalent to their heights as shown in article ya of the book of *Elements*.

Whenever we use the word "solid" we mean parallelepiped, by "a surface" we mean rectangle.

Introduction 3 (Lemma 3): The parallelepiped abcd is given, and its base ac is a square. We need to construct a parallelepiped whose base is a square and whose height equals the given ht, and whose volume equals that of abcd.

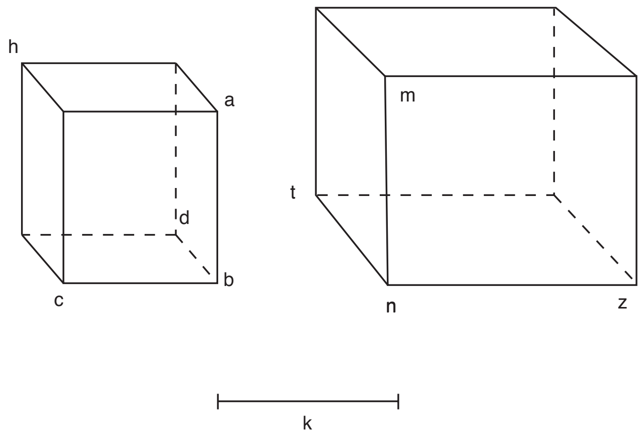
We make the ratio of ht to bd equal to the ratio of ab to k. Between ab and k, we construct a line which is middle in ratio between them, call it hz. We make hz perpendicular to ht, and complete the parallelepiped tz. Make mh perpendicular to the surface tz and equal to zh, and then complete the parallelepiped mhtz.

I claim that the parallelepiped t, whose base is the square mz and its height is the given ht, is equal (in volume) to the given parallelepiped d.

Proof: The ratio of the square ac to the square mz is the same as the ratio of ab to k. So the ratio of the square ac to the square mz is the same as the ratio of ht to bd. The bases of the two parallelepipeds are equivalent to their heights, so they are equal, and that is what we wanted to show.

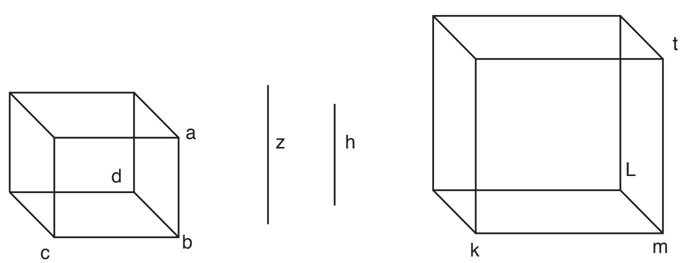
Now we discuss the third type of simple equations, which is a cube equals a number.

We set the number equal to the volume of a parallelepiped abcd whose base is ac, which is a unit square as we stated, and its height equals the given number. We want to construct a cube that is equal to it (in volume).



We construct two intermediate lines between the two lines ab and bd, whose ratios are in between too; so their magnitudes are known as we stated. The two lines are named h, z, and we make mt equal to the line h on which we construct the cube tmkl. Then, the volume of tmkl is known and the length of its side is known. I claim that its volume equals that of the parallelepiped d.

Proof: The ratio of the square of ac to the square of tk is the same as twice the ratio of ab to mk. Further, twice the ratio of ab to mk is the same as the ratio of ab to z (the first to the third of the four lines), which is the ratio of mk the second line to the fourth bd.



So the bases of the cube l and the parallelepiped d are equivalent to their heights, so they have the same volume. That is what we wanted to show.

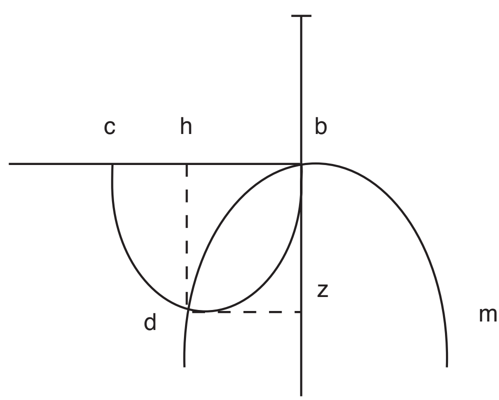
From this point on, we shall discuss the remaining six types of equations of three terms each.

First type: Cube plus sides (roots) equals a number ($x^3 + ax = b$)

We set ab to be a side of a square whose length equals the given number of the roots. Then we construct a parallelepiped with a square base whose side is ab and its height is bc, which we assume to equal the given number. The construction is similar to what we have done before. We make bc perpendicular to ab.

You know by now what we mean by a numerical parallelepiped. It means a parallelepiped whose base is a unit square and whose height is the given number (I mean a line whose ratio to the side of the base of the parallelepiped is the same as the ratio of the given number to one). We extend ab to z, then construct the parabola mbd, with vertex b, axis bz and its perpendicular side ab, so the parabola mbd is known as we have shown previously, and it is tangent to the line bc. We construct a semicircle on bc which must intersect the (conic) section, say at d.

From d, which is of known location, we draw two perpendicular lines, dz and dh, to the lines bz and bc respectively, so their magnitudes and locations are known. The line dz is one of the lines of order in the (conic) section, so its square equals the product of (the magnitudes of) bz and ab. Hence the ratio of ab to dz (which is equal to bh) is the same as the ratio of bh to hd (which equals zb). But the ratio of bh to hd is the same as the ratio of hd to hc. So the four lines ab, bh, hd and hc are proportional. Hence the ratio of the square of ab (the first) to the square of bh (the second) is the same as the ratio of bh (the second) to hc (the fourth). Hence, the parallelepiped, whose base is the square of ab (which is the number of the roots) and height is hb, equals the (volume of the) cube bh, because their heights are equivalent to their bases. We make the parallelepiped whose base is the square of ab and height hb, common, so the cube bh plus this parallelepiped equals the (volume of the) cube whose base is ab and height is bc, which we assumed to equal the given number.



But the parallelepiped whose base is the square of ab (which is the number of the roots) and whose height is bh, which is the side of the cube, equals the given number of the sides of the cube hb. Hence the cube hb plus the assumed number of its sides equals the given number, which is the desired result.

This type has no different cases and none of its problems are impossible to solve. Properties of the circle and the parabola were used.

Second type: Cube plus a number equals sides ($x^3 + a = bx$)

Assume ab is a side of a square that is equal (in magnitude) to the number of the roots. We construct a parallelepiped with a square base of side ab, and whose (volume) equals the given number. Let bc be its height which is perpendicular to ab. We construct the parabola dbh, with vertex b, axis along ab and its right side is ab. So the location of this parabola is known.

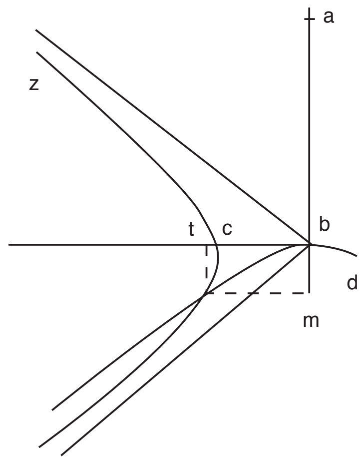
Now, we construct another (conic) section, the hyperbola hcz, with vertex at c, its axis along bc, and each of its two sides, the perpendicular and the oblique, equals bc. The location of this hyperbola is known, as was shown by Apollonius in proposition nh of article a. These two (conic) sections either meet or do not meet. If they do not meet, then there is no possible solution. If they meet tangentially at one point, or they intersect at two points, then the location of these point(s) is (are) known.

Assume they intersect at the point h. We draw two perpendiculars ht and hm to the two lines bt and bm. Surely, the location and magnitude of these perpendiculars are known. Since the line ht is one of the lines of order, it follows that the ratio of the square of ht to the product of bt and tc is the same as the ratio of the perpendicular side to the oblique one, as was shown by Apollonius in proposition k, article a. The two sides, the perpendicular and the oblique, are equal. Hence, the square of ht equals the product of bt and tc, and so the ratio of bt to th is the same as the ratio of th to tc. But the square of hm (which is equal to bt) equals the product of bm and ba, as was shown by proposition yb of article a in the book of conic sections.

So the ratio of ab to bt is the same as the ratio of bt to bm and is as the ratio of bm (which equals ht) to tc. Thus, the four lines are proportional. Hence, the ratio of the square of ab (the first) to the square of bt (the second) is the same as the ratio of bt (the second) to tc (the fourth). So the cube of bt equals (the volume of) the parallelepiped whose base is the square of ab and height equals ct.

We make the parallelepiped (whose base is the square of ab and height bc, which we constructed to equal the given number) common, so the cube bt plus the given number equals the parallelepiped whose base is the square of ab and height equals bt, which equals the number of the sides of the cube.

We found that this type has many different cases, some of which have no solution, and the properties of the two conic sections (hyperbola and parabola) were used.



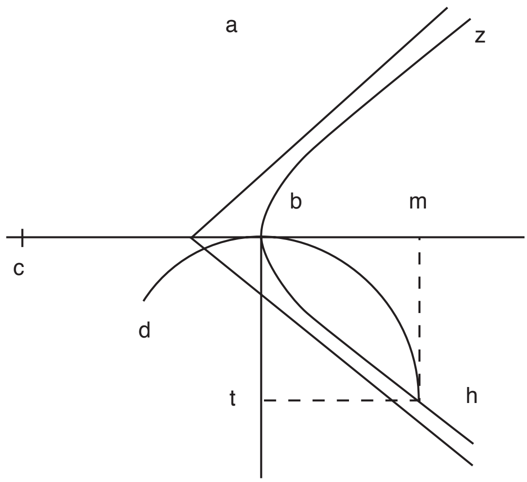
Third type: Cube equals sides plus a number ($x^3 = ax + b$)

We set ab to be a side of a square that (its area) equals the number of the sides. We construct a parallelepiped with base equal to the square ab, and (whose volume) equals the given number. Let its height bc be perpendicular to ab. We extend ab and bc, then we construct the parabola dbh, with vertex at b, axis along ab and its perpendicular side is ab, so its location is known. The conic section is tangent to bc, as was shown by Apollonius in proposition lc in article a. Then we construct another conic section, the hyperbola zbh, with vertex at b, axis along bc, and each of its sides, the perpendicular and the oblique, equals bc. So the location of this section is known and it is tangent to the line ab. The two conics must intersect. Assume they intersect at h, so the location of h is known. From the point h, we construct the two perpendiculars ht and hm, which are of known location and magnitude. The line hm is one of the lines of order. It follows from the above construction that the square of hm equals the product of cm and bm. Hence the ratio of cm to hm is the same as the ratio of hm to mb. But the ratio of hm (which equals bt) to mb (which equals ht, which is a line of order of the other section) is equal to the ratio of ht to ab (which is the right side of the conic section).

So the four lines are proportional. Hence the ratio of ab to mb is the same as the ratio of mb to bt, and is as the ratio of bt to cm.

Now, the ratio of the square of ab (the first) to the square of mb (the second) is the same as the ratio of mb (the second) to cm (the fourth).

Hence the cube of mb equals (the volume of) the parallelepiped whose base is the square of ab and height equals cm because their heights are equivalent to their bases. But this parallelepiped is equal (in volume) to the (volume of the) parallelepiped whose base is ab and height equals bc, which we constructed so that (its volume) equals the given number plus (the volume of) the parallelepiped, whose base equals the square of ab and height bm, which is equal to the assumed number of the sides of the cube bm. So the cube of bm equals the given number plus the given number of sides, which is the required result.



It is clear that there is only one case in this type, and the case of an impossible solution does not occur. The properties of the parabola and hyperbola were used.

Fourth type: Cube plus squares equal a number ($x^3 + ax^2 = b$)

We draw the line ab to equal the given number of squares, and we construct a cube that is equal (in volume) to the given number. Let m be its side. We extend ab and we set bt equal to m. Then we complete the square btdc. On d we construct the hyperbola hdn that does not meet bc and bt, as was shown in the two figures d and h of article b, and in figure nt of article a. The location of the conic section hdn is known because the point d is of known location, and the two lines bc and bt are of known locations.

We construct the parabola ak, with vertex at a, axis along at, and its right side is bc. So the location of ak is known.

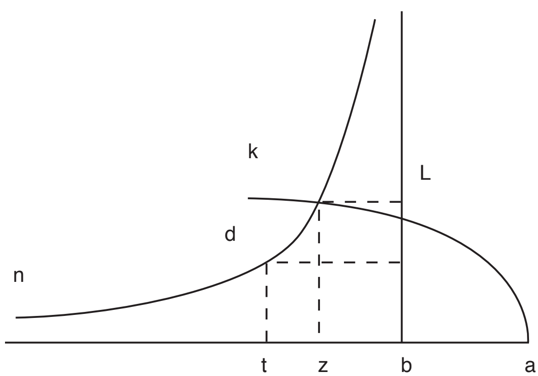
The two sections must intersect, say at h, so the location of h is known. From h, we construct the two lines hz and hl which are orthogonal to at and bc respectively, so the two lines are of known location and magnitude.

I claim that: it is not possible for the section ahk to intersect the section hdn at a point, where the perpendicular from that point to the line at meets the line at t or at a point on the extension through t.

Assume, if possible, it meets t, so its square equals the product of at and tb (which equals bc). But this perpendicular equals the perpendicular on dt, so the square of td equals the product of at and tb (which is also equal to the product of bt with itself), and that is impossible.

Also it does not meet the line at a point on the extension through t, because in such a case, that perpendicular must be shorter than td, which again is impossible. So the perpendicular must pass through a point between a and t, say z, and let the perpendicular be hz.

The square of hz equals the product of az and bc. Consequently, the ratio of az to hz is the same as hz to bc. Further, the rectangle hb equals the rectangle db (in area), as shown by proposition m of article b in the book of conic sections. Hence the ratio of hz to bc is the same as the ratio of bc to bz. Thus, the four lines az, hz, bc, bz are proportional. So the ratio of the square of bz (the fourth) to the square of bc (the third) is the same as the ratio of bc (the third) to az (the first). We conclude that the cube bc (which we construct so that it is equal in volume to the given number) equals (in volume) the (volume of the) parallelepiped whose base is the square bz and height az. But this parallelepiped whose base is the square bz and height az equals the cube bz, and equals the parallelepiped whose base is the square bz and height ab.



This parallelepiped (whose base is the square bz and height ab) equals the given number of squares. Consequently, the cube bz plus the given number of squares equals the given number, and that is what we wanted to show.

This type has no different cases and always has a solution. The properties of both the parabola and the hyperbola were used.

Fifth type: Cube plus a number equals squares ($x^3 + a = bx^2$)

Assume ac equals the given number of squares. We construct a cube that equals (in volume) the given number. Let its side be m.

The line m must be either equal to, greater than or less than ac. If they are equal, then the problem has no solution. This is because either the (length of the) side (root) of the required cube is equal to, less than or greater than m. If it is equal to m, then the product of ac and its square (the square of the required cube) equals the cube of m, so that the given number equals the number of the squares and there is no need to add the cube. If the (length of the) side of the cube is smaller than m, then the product of ac and its square will be smaller than the given number, so the number of squares is smaller than the given number without adding the cube. Finally, if the (length of the) side is larger than m, then its cube will be greater than the product of ac and its square without adding the number.

Further, if m is greater than ac, then it is even more clear that for the above three cases, there is no possible solution. So m must be less than ac or no solution will exist for the problem.

Cut a segment bc from ac of length equal to the length of m. The line bc is either equal to, less than or greater than ab. Let it be equal to bc, in the first figure, greater than in the second and less than in the third.

In the three figures, complete the square dc and construct a hyperbola which does not meet ac and ch. The section is dz in the first and dt in the second and third. We then construct a parabola with vertex a, axis along ac and its right side equals bc. It is at in the first, al in the second and ak in the third.

The positions of the two conic sections are known. In the first figure, the parabola will pass through the point d because the square of db equals the product of ab and bc, so d is on the parabola and they (the two conic sections) meet at some other point, as you may easily discover. In the second, the point d falls outside the parabola because the square of db is greater than the product of ab and bc. So, if the two sections meet tangentially at some other point, or if they intersect, then the perpendicular from this point must meet a point between a and b and a solution is possible; otherwise, no solution exists.

The honorable geometer Abu-Aljood was not aware of the case of intersection or tangency, which led him to claim that if bc was greater than ab, then the problem would be impossible. His claim was wrong.

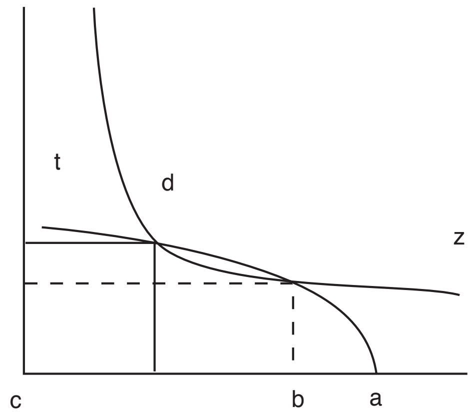
Al-Mahani was forced to study this one of the six known types.

In the third, the point d is inside the parabola, so the two conic sections will meet at two points. In summary, we draw from the point of intersection a perpendicular to ab, call it in the second figure zt. Another perpendicular (from the same point t) is drawn to ch, let it be tk. The rectangle tc equals the rectangle dc. So the ratio of zc to bc is the same as the ratio of bc to tz. The line zt is one of the lines of order in the conic section atl. Hence, its square is the same as the product of az and bc. Consequently, the ratio of bc to tz is the same as the ratio of zt to za. The four lines are proportional: the ratio of zc to cb is as the ratio of cb to zt and as the ratio of zt to za. So the ratio of the square of zc (the first) to the square of bc (the second) is the same as the ratio of bc (the second) to za (the fourth). Hence the cube of bc (which is equal to the given number) equals the parallelepiped whose base is the square zc and height za.

We make the cube zc common. So the cube zc plus the given number equals the parallelepiped whose base is the square zc and height ac, which equals the number of squares. This is the required result.

The other two cases are similar, except in the third case where we need to draw two cubes, because each perpendicular cuts one side of the cube from ca, as shown before.

It was shown that for this type of equation there are different cases, some of which have no solution. Properties of the two conic sections, hyperbola and parabola, were used.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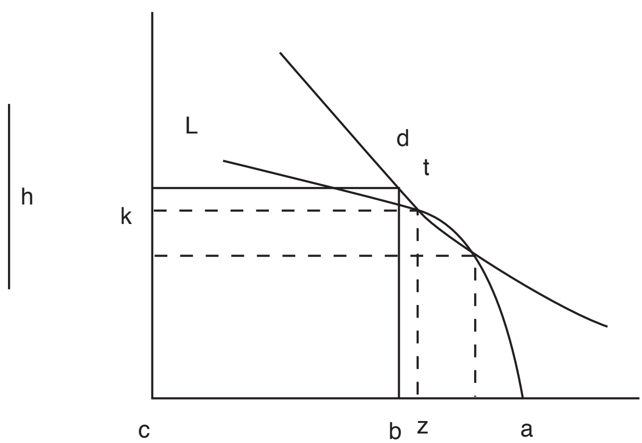
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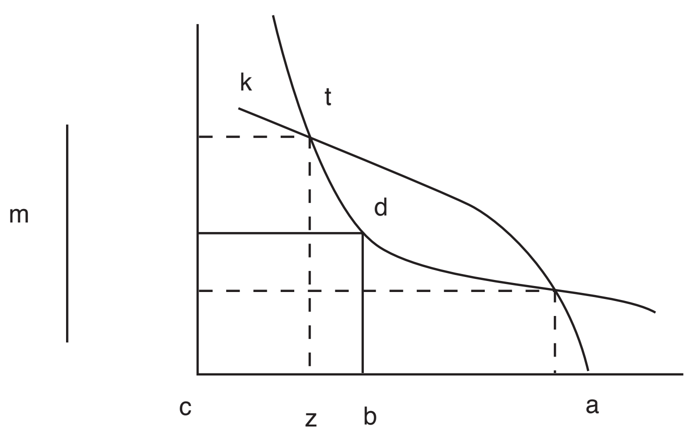
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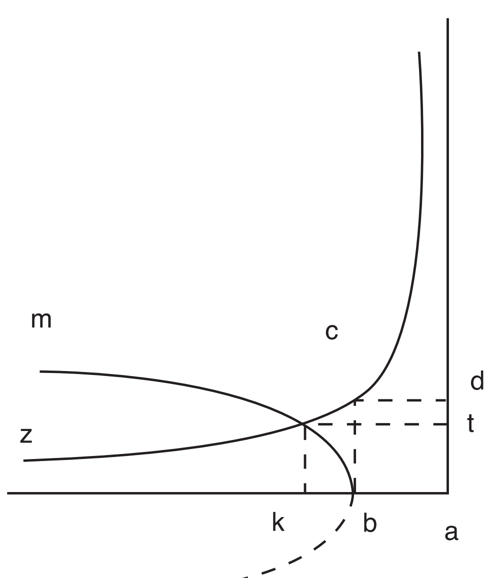
Sixth type: Cube equals squares and numbers ($x^3 = ax^2 + b$)

Assume the number of the squares equals the length of the line ab. We construct a parallelepiped with height ab and square base (whose volume) equals the given number. Let its side bc be perpendicular to ab. We then complete the surface (rectangle) db. We draw at the point c the hyperbola chz, which does not meet ab or ad. We draw another (conic) section, the parabola bhm, with vertex b, axis along ab, and right side ab. These two sections must intersect, say at h. So the location of h is known. We draw from h two perpendiculars, ht and hk, on ab and ad respectively. So the (area of) rectangle ha equals the rectangle ca. Hence the ratio of ak to bc is the same as the ratio of ab to hk, and their squares are also proportional. But the square of hk equals the product of kb and ab because hk is one of the lines of order of the (conic) section bhm. So the ratio of the square ab to the square hk is the same as the ratio of ab to bk. Hence, the ratio of the square of bc to the square of ak is the same as the ratio of bk to ab. Thus the parallelepiped whose base is the square bc and height ab equals (in volume) the parallelepiped whose base is the square ak and height kb because both of their bases and heights are equivalent. We make the parallelepiped with base the square ah and height ab common. So the cube ak equals the parallelepiped (whose base is the square bc and height ab which we constructed to equal the given number) plus the (volume) parallelepiped whose base is the square ak and height ab which equals the assumed number of squares. Consequently, the cube ak equals the given number of squares plus the given number.

This type has no different cases and a solution is always possible. The properties of the two (conic) sections, parabola and hyperbola, were used.



Third degree equations of four terms each, part 1

Since we have finished the study of equations of three terms each, we move now to the four types of four terms each; all have three terms equal to one term.

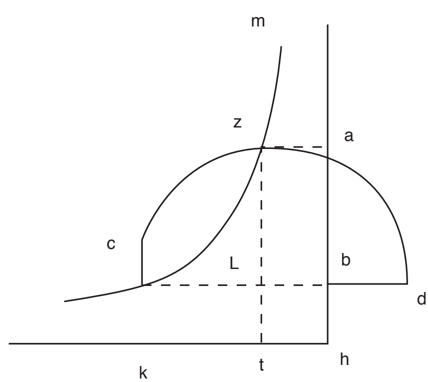
First type: Cube plus squares plus sides equals a number ($x^3 + ax^2 + bx = c$)

We draw bh a side of a square which is equal to the assumed number of sides. Then, we construct a parallelepiped whose base is the square bh and (whose volume) equals the given number. Let its height bc be perpendicular to bh. We construct bd along bc, to equal the given number of squares. On the diameter dc we construct the semicircle dzc, and we complete the rectangle bk. Then, we construct a hyperbola with vertex at c which does not meet bh or hk. So the parabola must intersect the circle at a point c, because it intersects its tangent line ck. Hence the section must intersect the circle at a second point, say at z. So the location of z is known because the locations of both the circle and the (conic) section are known.

From the point z we draw two perpendiculars, zt and za, to hk and ha respectively. The rectangle zh equals the rectangle bk. Removing the common rectangle hl, we are left with two equal rectangles, zb and lk. So the ratio of zl to lc is the same as the ratio of hb to bl because hb equals tl; further, their squares are also proportional. But the ratio of the square of zl to the square of lc is the same as the ratio of dl to lc – for the circle – so the ratio of the square of hb to the square of bl is the same as the ratio of dl to lc. Hence, the parallelepiped (whose base is the square hb and height lc) equals (in volume) the parallelepiped whose base is the square bl and height dl. But the second parallelepiped equals (in volume) the cube bl together with the parallelepiped whose base is the square of bl and height bd, which equals the given number of squares.

We make the parallelepiped (whose base is the square of hb and height equals bl, which equals the number of the roots) common. So the parallelepiped whose base is the square bh and height bc, which we constructed to equal the given number, will equal the cube of bl plus the number of the given sides plus the given number of squares. That is what we wanted to show.

This type has no different cases and always has a solution. Properties of circles and hyperbolas were used.



Second type: Cube plus squares plus numbers equals sides ($x^3 + ax^2 + b = cx$)

Construct ab to be the side of a square which is equal to the number of sides and bc to equal the given number of squares; further, bc is perpendicular to ab. We construct a parallelepiped whose base is the square with ab and is equal (in volume) to the given number. Let its height bd be along the extension of bc. At the point d, we construct the hyperbola zdm that does not meet ab or ah after we complete the rectangle bh. We construct another hyperbola, tdm, with vertex d, axis along the extension of bd and each of its two sides, the normal and the oblique, equals dc. No doubt, this section must intersect the first at d. If they meet at a second point, then the problem has a solution; otherwise it is impossible to solve the problem. Such a meeting of the two sections, whether by tangency at one point or by intersection at two points, is based on article four of the book of conics. We promised to refer to only two articles from that book. But once they meet, it does not matter how they meet, whether by tangency or by intersection. If they intersect at a point other than d, they must intersect at two points. In either case (tangency or intersection), we draw from the point m (the point of intersection or the point of meeting, as the case may be) two perpendiculars, mn and kml. Since the point m is of known location, both perpendiculars are of known location and magnitude.

The rectangle am equals (in area) the rectangle ad. We remove the common rectangle hn, and we get nd equals hm. We make dm common. It follows that nl equals hl because their sides are equal and so are the squares of their sides. Hence, the ratio of the square of ab to the square of bl is the same as the ratio of the square of ml to the square of ld.

But the ratio of the square of ml to the square of ld is the same as the ratio of cl to ld as we have shown repeatedly. Consequently, the ratio of the square of ab to the square of bl is the same as the ratio of cl to ld. Thus, the parallelepiped – whose height is ld and base equals the square of ab – equals the parallelepiped whose base is bl and height lc. But the latter parallelepiped equals the cube of bl plus the parallelepiped whose base is the square of bl and height bc, which equals the given number of squares.

We make the parallelepiped (whose base is the square of ab and height bd, which was constructed to equal the given number) common. Hence, the cube bl plus the number of squares plus the given number, equals the parallelepiped whose base is the square of ab and height bl (which equals the given number of sides of the cube of bl). This is the required result.

It is clear that this type has many different cases, including the case of two different roots for two different cubes. There might be cases that are impossible to solve – I mean the problem associated with the case is impossible to solve. Properties of two hyperbolas were used.

Third type: Cube plus sides plus numbers equals squares ($x^3 + ax + b = ex^2$)

We draw the line bh to equal the given number of squares and bc, a side of a square that is equal to the assumed number of sides, perpendicular to bh. We construct a parallelepiped whose base is the square of bc and is equal (in volume) to the given number. Let its height be ab on the extension of bh.

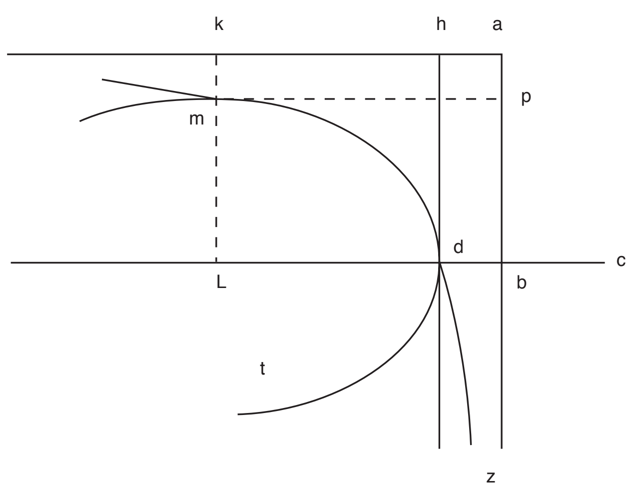
On ah we construct a semicircle azh. The point c is either inside the circle, on its circumference, or outside it. Let it fall first inside the circle.

We extend bc until it intersects the circle at z, then we complete the rectangle ac. Then, on zc, we construct the rectangle zm that is equal (in area) to the rectangle ac. The location of m is known because the position, magnitude and angles of the surface cm are known, and the location and the magnitude of the line zc are known. The point m must be inside the circle, on the circumference, or outside it.

Assume first that it falls inside the circle. We construct a hyperbola with vertex m, which does not meet zc and zn. Hence it must intersect the circle at two points, say l, n, whose locations are known. From these two points we draw the two lines lk and nf perpendicular to ah. Further, from l we draw the perpendicular lt to bz.

The rectangle lc equals (in area) the rectangle cm, and the rectangle cm equals ca. We make ck common, so dk equals tk, because their sides are equivalent and so are the squares of their sides. But the ratio of the square of lk to the square of ka is the same as the ratio of hk to kl. It follows that the ratio of the square of bc to the square of bk equals the ratio of hk to ka. Hence, the parallelepiped whose base is the square bc and height ka equals (in volume) the parallelepiped whose base is the square of bk and height kh. But the first parallelepiped equals the given number of sides of the cube bk and equals the given number. We make the cube bk common. Consequently, the parallelepiped (whose base is the square bk and height bh, which equals the given number of squares of the sides of the cube bk) equals the cube bk plus the given number of sides plus the given number.

A similar result holds for the cube bf, if the points c and m fall inside the circle.



If m falls outside the circle, then we construct the hyperbola with vertex at m. The section either intersects the circle or it meets the circle tangentially. In either case, it will be similar to what we have already mentioned.

The variety of this type was mentioned by Abu-Aljood in solving the problem which we will discuss.

If the section does not meet the circle, we construct a surface on a line shorter than zc or longer, when m lies inside the circle. If the section in such a case does not meet the circle then no solution is possible. The proof that no solution is possible is the opposite of what we have mentioned.

If c falls on the circumference of the circle or outside it, then we extend cz and construct a rectangle, with one of its angles at c, in such a way that if we construct at the opposite angle to c a section as described before, it will meet the circle by intersection or tangency. This can be proved by a simple deduction that I have left as a mathematical exercise for the reader of my essay. Whoever could not prove the deduction will gain nothing from this essay, it being based on the previously mentioned three books. We prove that the impossible is not possible by reversing the proof that we presented for the possible. This is based on the fact that the side of the cube must be shorter than bh, which equals the given number of squares. For if the side of the cube is equal to the assumed number of squares, then the cube is equal (in volume) to the assumed number of squares plus some part of the number and the sides.

If the side of the cube is larger than the given number of squares, then the cube itself will be larger than the given number of squares plus something else added to it. So the side of the cube has to be smaller than hb.

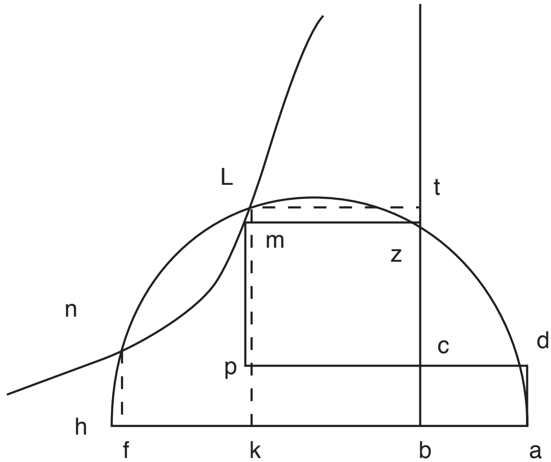
We separate a part from bh that is equal to the side of the cube, let it be bf. At f, we construct a perpendicular that meets the circumference of the circle, then we reverse the above-mentioned proof. This shows that the tip of the perpendicular is on the circumference of the conic section, whereas we said it does not meet the circle, and this is impossible.

Since we believe that this deduction might be difficult for some readers of this essay, we leave this argument and present a formula instead of this deduction.

We construct, in any way we want, a rectangle on the line of extension of bc regardless of the location of c, inside or outside the circle. We make such a rectangle so that one of its angles is at c and it is equal to the rectangle ac. Certainly, the sides of this rectangle are of known location and magnitude.

On the opposite angle of c, we construct a hyperbola that does not meet zc and cn (which is perpendicular to zc at c). If the section meets the circle by intersection or tangentially, then a solution is possible, otherwise it is impossible. The proof of the impossibility is as we have already mentioned.

One of the geometers needed this type in his work, and succeeded in proving it, but he did not show the different cases that may occur and he did not realize that there is a case where the solution is impossible as we have mentioned. So make sure you understand this type and the last formula in the proof of this type together with the distinction between the possible and the impossible cases.



This type uses the properties of the circle and the hyperbola, and this is what we needed to show. But the problem that led one of the late geometers to this type is: divide ten into two parts so that the sum of the squares of the two parts plus the outcome of dividing the larger part by the smaller one equals seventy-two.

He denoted one part by x, and the other by 10 – x, as algebraists usually do in such divisions. This division leads to: cube plus five plus thirteen and a half of the side of the cube equals ten squares. In this specific problem, the two points c and m will fall inside the circle.

This honorable man was able to solve this problem, though a number of honorable men from Iraq could not solve it, among them was Abu Sahel Al-Qohi. But this man (who solved the problem) despite his great knowledge of mathematics, did not realize all the different cases, and that some cases are impossible to solve. This honorable man is Abu-Aljood, nick-named Al-Shanni, and God knows better.

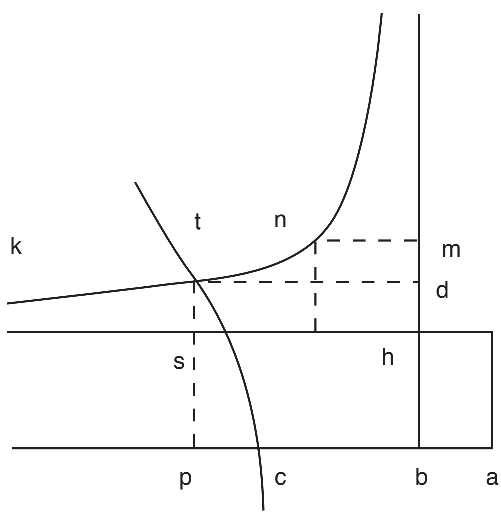
Fourth type: Numbers plus sides plus squares equals a cube ($a + bx + cx^2 = x^3$)

Assume bh to be a side of a square that is equal to the assumed numbers of sides. Construct a parallelepiped whose base is the square bh and (whose volume) equals the given number. Let its height ab be perpendicular to bh. We let bc be along ab and is equal to the assumed number of squares. Then we complete ah and we extend hb along hm regardless of the magnitude. On the given hm we construct the surface hn that equals ah. So the location of the point n is known. At n, we construct the hyperbola ntk that does not meet hm and hs, and so it is of known location.

We construct another hyperbola, lct, with vertex at c, and axis along bc, and each of its sides, the perpendicular and the oblique, equals ac. So the location of this hyperbola is known. Further, it must meet the section ntk, say at t, so t is of known location.

From t, we construct the two perpendiculars td and tp to bc and bm. The locations and magnitudes of such perpendiculars are known. Further, th equals hn which in turn equals ha. We make hp common, so as equals tb because their sides are equal (in length) and so are the squares of their sides. But the ratio of the square of tp to the square of ap is the same as the ratio of pc to ap, as we have shown many times for the section lct.

Now, the ratio of the square of bh to the square of bp is the same as the ratio of pc to pa. Thus, the parallelepiped whose base is the square bh and height ap equals the parallelepiped whose base is the square bp and height cp. But the first parallelepiped equals the parallelepiped which we constructed to be equal to the assumed number plus the parallelepiped whose base is the square bh and height bp (which is equal to the given number of the sides of the cube bp).



We make the parallelepiped, whose base is the square bp and height bc (which is equal to the given number of squares of the cube bp) common. So the cube of bp is equal to the given number of squares plus the given number of sides plus the given number. This is the needed result. This type has no different cases and always has a solution.

Third degree equations of four terms each, part 2

Since we finished the four types each of which consists of four terms, we move on to the three types each of which consist of two terms equals two terms.

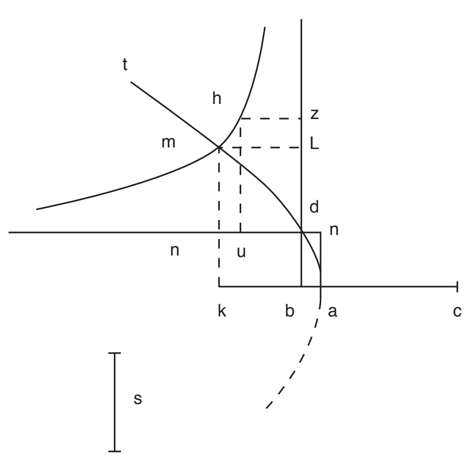
First type: Cube and squares equal sides and numbers

We make bd a side of a square which equals (in area) the assumed number of sides, and construct cb to equal the assumed number of squares and be perpendicular to bd. We construct a parallelepiped with base bd and volume equal to the assumed number; let its height be s. So the line s is either greater than, smaller than or equal to bc.

First, let it be smaller. We cut from bc the segment ab that equals s, and we complete the square ad. Assume dz is an extension of bd regardless of magnitude. We construct a rectangle on dz that is equal to ad, call it hd. So the location of h is known and the sides of the rectangle hd are all of known location and magnitude. At h, we construct the hyperbola hm, that does not meet zd or dy. So the location of hm is known. We construct another hyperbola amt, with vertex at the point a and axis ab, and each of its sides, the perpendicular and the oblique, equals ac. This hyperbola must intersect the other section. Let the two sections intersect at m, so the location of m is known. From m we draw two perpendiculars mk and ml whose location and magnitude are known. The rectangle md equals the rectangle hd which in turn equals the rectangle ad, and dk is common. The surface (rectangle) hb equals (in area) the rectangle am; so their sides are equivalent and hence the squares of their sides are equivalent. But the ratio of the square of mk to the square of ka is the same as the ratio of ck to ak, as we have shown many times. Hence, the ratio of the square of bd to the square of kb is the same as the ratio of ck to ak. The parallelepiped, whose base is the square of bd and height ak, is the same as the parallelepiped whose base is the square of bk and height ck. But the latter parallelepiped equals (in volume) the cube bk plus the parallelepiped whose base is bk and height bc which equals the given number of squares.

The first parallelepiped equals the parallelepiped whose base is the square of bd and height ab, which we constructed to equal the given number plus the (volume) parallelepiped whose base is the square bd and height bk, which equals the given number of sides of the cube bk.

So the cube of bk plus the given number of its squares equals the given number plus the given number of sides. This is the required result.



If s equals bc, then bd is the side of the required cube.

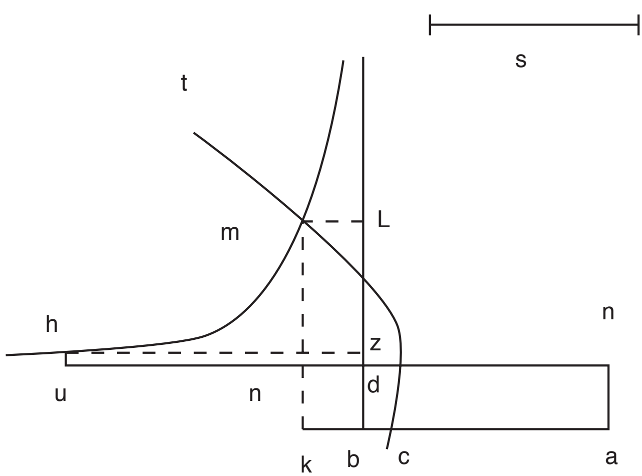
Proof: The solid (parallelepiped) whose base is the square of bd, and whose height also equals bd (which equals the number of sides of the cube bd) equals (in volume) the cube of bd.

The cube, whose base is the square of bd and whose height is bc (which equals the given number of squares of the cube of bd), equals the parallelepiped whose base is the square of bd and height s (which equals the assumed number). So the cube of bd plus the given number of squares equals the assumed number plus the assumed number of sides, and this is the desired result.

In such a case, it is known that the cube of bd plus the assumed number equals the assumed number of squares (of the cube of bd) plus the given number of sides. There is an intersection between this type and the third type, which is: cube and numbers equal squares and sides.

If s is greater than bc, then we make ab equal to s, and construct the second conic section at the point c such that each of its two sides equals ac. This second section must intersect the other section, and the side of the cube will again be bk. The rest of the construction and proof is similar to the previous one except that the ratio of the square of mk to the square of ka is the same as the ratio of ak to kc.

We found that this type has different cases and types; one of its two cases has much in common with the third type and each of the cases has a solution, and it uses the properties of two conic sections.



Second type: Cube plus sides equals squares and numbers

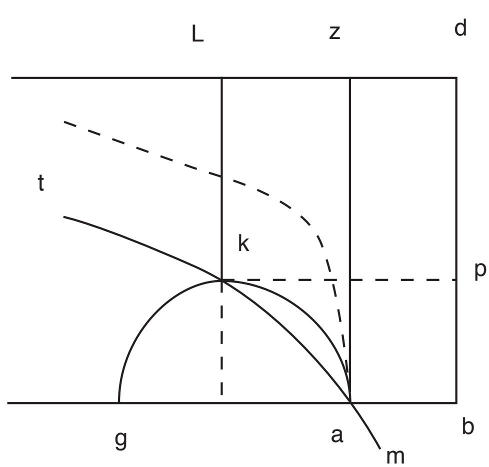
We set bc to equal the number of the given squares. We draw bd to be a side of a square that is equal (in area) to the number of sides, and is perpendicular to bc. Then, we construct a parallelepiped which is equal (in volume) to the given number, such that its base is the square of bd. Let its height be s. The line s is either smaller than, equal to or larger than bc.

Assume first that s is smaller than bc. We cut from bc the segment ba which equals s. Then we form the rectangle ad, and construct the circle akc, with diameter ac. The location of the circle is known. At the point a, we construct a hyperbola which does not meet bd or dz, call it mat. Again its location is known and mat will intersect az, the tangent to the circle. So it must intersect the circle, because if it falls between the circle and az we could draw from the point a a line which is tangent to the section as was shown by Apollonius in diagram s in his article b. That line either falls between az and the circle, which is impossible, or it falls outside az, so az would be a straight line between the conic section and the tangent line, and that is impossible too. So the conic section tam does not fall between the circle and az. So it must intersect the circle, and it must intersect it at another point, call it k, whose location is known. We draw two perpendiculars from k, say kn and kh to bc and bd. Both the location and magnitude of kn and kh are known. We complete the rectangle kd.

The rectangle ad equals the rectangle kd. We remove the common square nz and we make ak common. So bk equals al since their sides are equivalent, and hence the squares of their sides are equivalent. But the ratio of the square of kh to the square of ha is the same as the ratio of hc to ha. It follows that the ratio of the square of bd to the square of bh is the same as the ratio of ch to ha. Hence, the parallelepiped whose base is the square of bd and height ha, equals (in volume) the parallelepiped whose base is the square of bh and height ch.

We make the cube bh common, so the parallelepiped whose base is the square of bh and height bc, equals the cube of bh plus the volume of the parallelepiped whose base is the square of bd and height ha. But the first parallelepiped equals the given number of squares of the cube bh.

We set the parallelepiped – whose base is the square bd and height ba which we constructed to equal the given number – common. So the cube bh plus the (volume) parallelepiped – whose base is the square bd and height bh which equals the given number of sides of the cube bh – equals the given number of squares plus the given number, and this is the desired result.



If s equals bc, then bc is the side of the required cube.

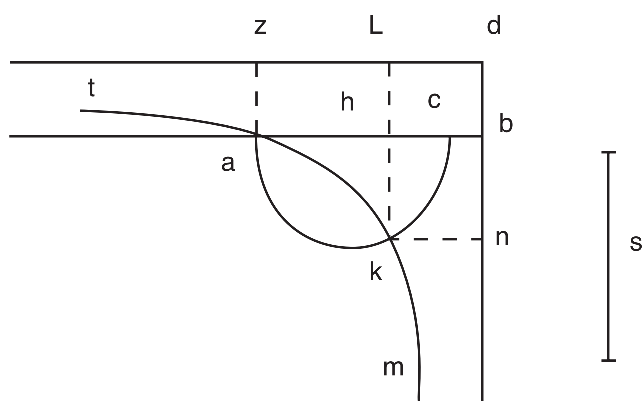
Proof: The cube of bc equals the given number of its squares. And the parallelepiped, whose height is bc and whose base is the square of bd, equals the given number, and it also equals the assumed number of the sides of the cube bc. So the cube bc plus the given number of its sides equals the given number of squares plus the given number.

This type has intersection with the third kind, because the given number of the sides of the cube bc equals the assumed number. It follows that the cube of bc plus the given number equals the given number of its squares plus the given number of sides.

If s is larger than bc, we set ba equal to s and construct the circle with diameter ac, and the conic section at a intersects the circle at k as we have shown. From k we draw two perpendiculars kh and kn as we did in the previous figure, so bh is the side of the required cube. The proof of this claim is as before.

We omit the common rectangle hd, so the sides hn and hz are equivalent and their squares are equivalent. The rest of the proof is exactly as the previous one.

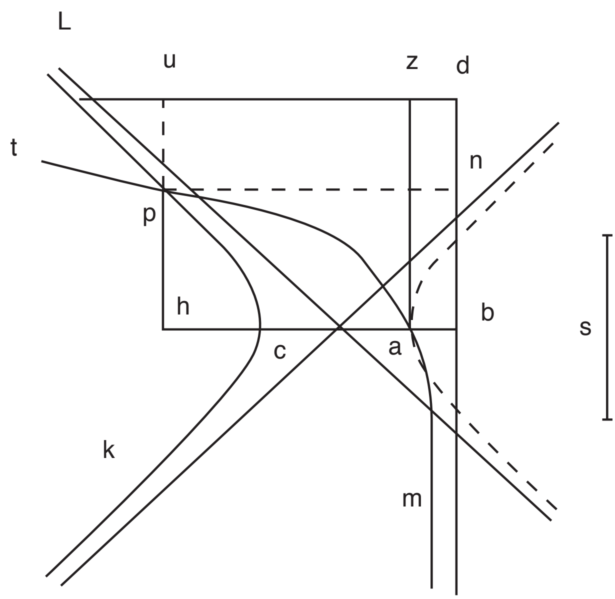
We saw that this kind has different cases and types. One of them has some intersection with the third type and none of its cases are impossible to solve. The properties of circles and hyperbolas were used.



Third type: Cube and numbers equals sides and squares

Assume that bc equals the assumed number of squares. Let bd be perpendicular to bc and a side of a square that equals the assumed number of sides. Construct a parallelepiped whose base is the square of bd and is equal to the given number. Let s be its height. So the line s is smaller than, equal to, or larger than bc.

First, let it be smaller. We cut from bc the segment ba that equals s, and then we complete the rectangle bz. At a, we construct the hyperbola mat, which does not meet bd or dz. We construct another hyperbola, call it kcl, with vertex at c, axis along the extension of bc, and each of its two sides, the perpendicular and the oblique, equals ac. So it must intersect the other section. Assume that the section kcl and the section mat intersect at p. So the location of p is known because the two sections are of known locations. From p we construct two perpendiculars pn and hpu, whose locations and magnitudes are known. The rectangle da equals the rectangle dp, so nh equals zh as we have shown many times. Their sides are equivalent and so are the squares of their sides. But the ratio of the square of ph to the square ha is the same as the ratio ch to ha. So the ratio of the square of bd to the square of bh is the same as the ratio ch to ha. The parallelepiped whose base is the square of bd and height ha equals (in volume) the parallelepiped whose base is the square of bh and height ch.

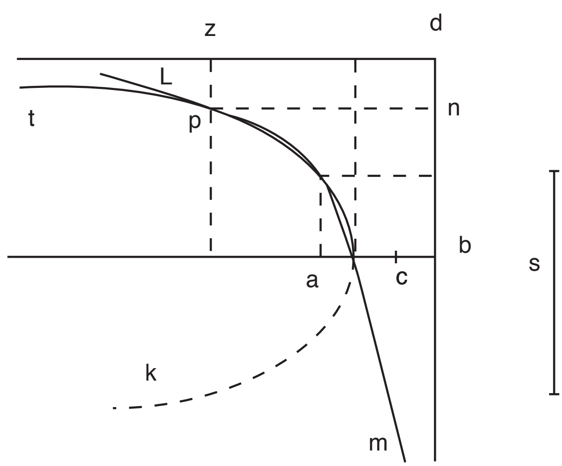


We make the parallelepiped – whose base is the square of bh and height bc, which equals the number of the squares of the cube bh – common. The cube of bh equals the assumed number of the given squares plus the (volume) parallelepiped whose base is the square of bd and height ha. We then make the parallelepiped – whose height is ba and base the square of bd which we constructed to equal the given number – common. Hence, the parallelepiped – whose base is the square of bd and height bh, which equals the number of the sides of the cube of bh plus the number of the squares of the cube bh – equals the cube of bh plus the given number.

If s equals bc then bc is the side of the (required) cube.

Proof: The cube of bc equals the assumed number of its squares. The assumed number equals the assumed number of the sides of the cube of bc. So the cube of bc plus the given number equals the given number of its squares plus the given number of its sides, which is the claim. Also, the cube of bc plus the given number of its sides equals the given number of its squares plus the given number. This type has some intersection with the second kind.

If s is larger than bc, we make ba equal to s and then complete the rectangle. On a, we construct the two sections (the first and the second). They must intersect. They meet at one point if they are tangent to each other, or at two points if they intersect, as known from article d of the book of conic sections. The problem either has a solution or not. If the conic sections intersect, we draw from the two points of intersection two perpendiculars that separate two sides of two cubes. The proof is as the previous one with no change. This type has many kinds; some have no solution and the properties of two conic sections were used.



And it was clear that the three quartile types have common cases. That is to say, there is a kind of the first that is exactly a kind of the second, and a kind of the second that is a kind of the third, and a kind of the third that is exactly the same as a kind of the second, as shown.

Equations that contain the inverse of the unknown

We have studied the twenty-five types of the introductory algebra and equations and covered them completely. We discussed the different kinds of each type. We also presented rules of how to distinguish the ones with possible solutions from those which are impossible to solve. We showed that most of them have solutions. Let us now study their parts (inverses).

The part of a number (its reciprocal) is a number whose ratio to one is the same as the ratio of one to the number. So if the number is three, then its part is one third, and if the number is one third, than its part is three. Similarly, if the number is four, then its part is one fourth, and if it is one fourth then its part is four. In summary, the part of a number is the part called by the same name as the number, such as one third to three if the number is an integer, and three to one third if it is a rational number. Similarly, the part of a square is the part called by the same name whether it be an integer or a rational. Likewise for the part of a cube. To make things more clear, let us put it in a table.

Part of root	**Part of square**	**Part of cube**
1	1	1
2	4	8

One	**Root**	**Square**	**Cube**
1	2	4	8

So the ratio of the part of the cube to the part of the square equals the ratio of the part of the square to the part of the root; and also equals the ratio of the part of the root to one, and equals the ratio of one to the root; and equals the ratio of the root to the square and equals the ratio of the square to the cube. These are seven ranks in a sequence, with the same ratio. We will talk about their equations only.

The part of square of the square and the part of square of the cube and the part of cube of the cube, as far as we can go, will also be in ratio and we do not need to mention them, for there is no way to deduce it.

Let it be known that if you take one eighth – which is part of cube – as a cube then its part is eight, which is a cube and vice versa.

Take this as a model for the rest. The part of the cube and the part of the square and the part of the root and one; these four have the same status as the cube, square, root and one.

Example: If we say: part of a square equals half part of a root, then this is equivalent to saying: a square equals twice a root. So the square is one quarter which is a part of a square. Hence, the required square is four and its part is one fourth and the part of its root is half and so on.

As for the complex ones (multi-term ones): If one says: part of a square plus two parts of a root equals one plus one fourth, then this is equivalent to: a square plus two roots equals one plus one fourth. Using the same method we used for the simple cases above, we get the root equals half and the square equals one fourth. But according to the question: part of a square plus two parts of a root equals one plus one fourth, we get the fourth (which is the first square) is the part of the required square, and so the required square is four.

Also, in quartiles (equations with four terms): If one says: part of a cube plus three parts of a square plus five parts of a root equals three plus three eighths, this is the same as saying: a cube plus three squares plus five roots equals three and three eighths.

Using the method that we explained via conic sections, we can determine the side of a cube, which will be the part of the required root. We set its ratio to the assumed one to equal the ratio of the assumed one to another line, so such a line is the side of the required cube. So it seems that there are twenty-five more types of these equations among these four, comparable with the previous twenty-five types.

As for multiplying them with each other, this is discussed in the books of algebraists, and one can deduce it easily, so there is no need to go into details. As for relating these four (equations) with the previous four, I say: If we say: a cube equals ten parts of a cube (that is: ten parts of itself) then the cube would be the first in seven ranks, and the parts of the cube are the seventh of them. So multiply one by the other and take the root of the outcome; the result is the middle term, I mean the fourth, which is the required cube.

To see this: Notice that if any number is multiplied by its reciprocal, the result is one. If it is multiplied by twice its reciprocal, the result is two. And if it is multiplied by ten times its reciprocal, the result is ten times the number. This is equivalent to the question: "what cube when multiplied by itself gives the product ten?" So its root is the required cube, and finding the side of that cube is done as we have shown using conic sections.

Similarly, the question: "what square equals sixteen of its reciprocal?" To solve it, multiply the reciprocal by sixteen and take the root of the outcome, which equals four, that would be the required square. This is equivalent to the question: "what square when multiplied by itself equals sixteen?"

Similarly, the question: "what root equals four of its reciprocal?" is equivalent to the question: "what number multiplied it by itself gives four?", and the answer as we know is two.

However, a deduction of a solution to the question: "what square equals a number of parts of the side of its cube?" is not possible using the previous methods. This is because we need to introduce four lines between two lines, in such a way that the respective ratios (of their lengths) of the six lines are the same. This was deduced by Abu Ali Ibn Al-Haitham, but it is very difficult and cannot be included in this book.

Similarly for the question: "what cube equals a number of parts of the square of its side?" It needs the above construction which could not be solved by our methods.

In summary, multiplying the first by the sixth of these seven ranks needs the introduction of four lines between two lines so that the respective ratios (of their lengths) of the six lines are the same, as was shown by Abu Ali Ibn Al-Haitham.

But to solve the question: "what cube equals sixteen parts of its side?" we multiply the first by the fifth, so the root of the root of the number equals the side of the required cube. The same method is used for any of the seven ranks that equals its fifth in the ratio within the seven ranks.

In the complex ones (multi-term equations), the statement: "a root equals one and two parts of the root", is equivalent to: "a square equals a root and two". This is because these three are in ratio with the previous mentioned three, and we solve it using the above-mentioned method. We get the square equals four which equals its root and two. The root of this is the needed number. Its root is two, which equals one and two parts of its root.

Similarly, "a square plus its two roots equal one plus two parts of a root", is equivalent to "a cube and two squares equal a root and two". We find the side of the cube using the method of conic sections, so the square of that side is the needed square.

Also, "a root plus two plus ten parts of the root equals twenty parts of a square" is equivalent to "a cube and two squares and ten roots equal twenty". So we get the side of the cube using conic sections and it will be the required root.

In summary, any four successive ranks of these seven ranks would fall in the same category of the mentioned twenty-five types. If it exceeds five, six or seven ranks, then there is no way to solve it. For example, the equation: a square plus two roots equals two plus two parts of the square, cannot be solved, because the square is the second rank and the part of the square is the sixth, and that exceeds five ranks. We use the same ideas for the other cases.

The odd types among these seven ranks amount to twenty-one types, two of which cannot be solved using our methods, and they need the argument of Ibn Al-Haitham. So, we are left with nineteen types that can be solved using our methods. We need to use the properties of the circle to solve some, and the properties of conic sections for others.

All the sequential triplicates constitute fifteen types, and they can be solved using the properties of the circle. The triplicate ones in every four consecutive ranks constitute twenty-four types, and can be solved using conic sections. Further, all the quartiles among every four consecutive ranks constitute twenty-eight types, and can be solved using properties of conic sections.

So all the types (which fall within these seven ranks that can be solved using our methods) amount to eighty-six; only six were mentioned in the books of ancient algebraists.

Anyone who read these introductions and had incentive and insight into these problems could have discovered what the ancients could not. It is time to end this essay with gratitude to God, and praising all of His prophets.

Abu-Aljood Ben Al-Laith problem

Five years after I finished this article, I was told by some people who knew little of geometry, that the geometer, Abu-Aljood Mohammad Ben Al-Laith, has done some work on the classification of these types; that he solved most of them using conic sections, but did not cover all types and did not classify them as solvable ones and the ones with no possible solution; he only discussed minor problems.

I thought that might be possible because these two types (of equations) which I attributed to someone else, are attributed to him. And I read them in a collection of classifications for Abu-Aljood with the handwriting of Al-Hazimi Al-Khawarazmi, including a three-term equation: cube plus a number equals squares. It has different kinds, with certain conditions on each kind, as was mentioned in this article. Some of these conditions were not fulfilled in his setup. He claimed that there is no solution of this kind, saying: if the side of the cube that equals the assumed number is larger than half the number of the squares, then the problem has no solution, which is not true as we have shown. The reason (behind his false claim) is that he was not aware of the case that the two conic sections may meet tangentially or by intersection.

A second type (of equations) that Abu-Aljood discussed is a quartile (four-term equation): a cube plus a number plus sides equals squares. I acknowledge that he studied this problem thoroughly after many geometers failed to solve it, though the problem was a minor one.

This type has different kinds and there are conditions to be fulfilled. Some of its (this type) problems have no solution, and Abu-Aljood did not study them thoroughly. I mention all of this so that any person who has a chance to read both articles could compare my article with this honorable person's article and see if what I was told about this honorable person is correct.

I believe I did my best to study all problems thoroughly but with short proofs and without unnecessary details.

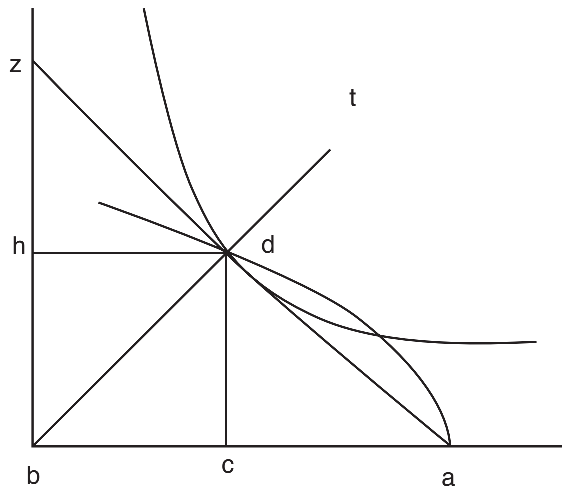
I could have presented an example for each of the types and kinds (of problems) to prove validity. But I avoided lengthy arguments and I limited myself to general rules counting on the intelligence of the student, because any person who has enough intelligence to understand this article would not fail to produce what he needs of partial examples. God gives guidance to the good, and on Him we depend.

One of my friends suggested that I should show the flaw in the proof of Abu-Aljood Mohammad Ben Al-Laith of the fifth type of the six triplicate (three terms) types that can be solved by conic sections which is: a cube plus a number equals squares.

Abu-Aljood said: We set the line $\underline{ab}$ to equal the number of squares, and we cut from $\underline{ab}$ the segment $\underline{bc}$ to be a side of a cube that equals the assumed number. So the line $\underline{bc}$ is either equal to, greater than, or smaller than $\underline{ca}$.

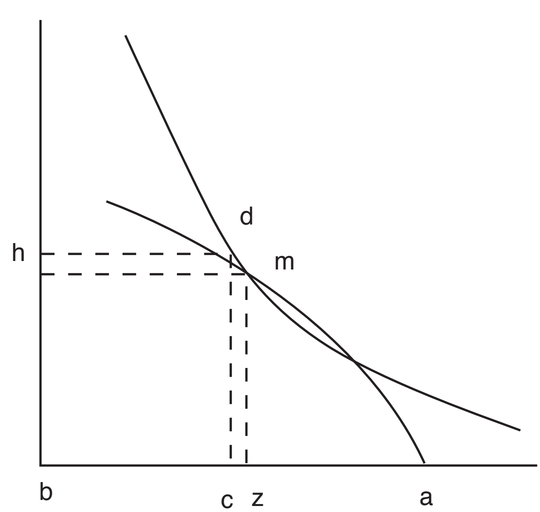
He said: If they are equal, we complete the rectangle $\underline{ch}$ then construct a hyperbola at $\underline{d}$ that does not meet $\underline{ab}$ and $\underline{bh}$. Further we construct a parabola with vertex at $\underline{a}$ and axis $\underline{ab}$ and right side $\underline{bc}$. This conic section will pass through the point $\underline{d}$ as we have shown. He claimed that the two sections will be tangent to each other at $\underline{d}$. This is false, because they must intersect.

Proof: We make $\underline{bz}$ equal to $\underline{ba}$, then draw $\underline{az}$. It must pass through $\underline{d}$.



Further, it will be inside the parabola, the angle adb will be a right angle and the two angles abd and zbd are equal. It is known that the axis of a parabola divides the angle of the conic into halves. So the line bdt must be the axis of the hyperbola at d, and the line ad is parallel to the lines of order, so it is tangent to the hyperbola.

Consequently, the parabola must intersect the hyperbola and not be a tangent to it, for if it were a tangent, then the lines from d to any point on the part ad of the circumference of the parabola would be between the conic and its tangent, which is impossible. So the parabola must intersect the hyperbola at d and at another point, say a, and that is the required result.



This was the mistake that the honorable man made when he said that the two conics must be tangent to each other at d.

His claim: "If bc was larger than ac, then the problem has no solution because the two conic sections would not meet", is a false claim. They could meet at one or two points or be tangent to each other between a and d as we have shown earlier. There is a more general proof than the one we presented:

Let the number of the squares be ab and bc be the side of the cube, which is more than half of ab. Complete ch and then construct the two conic sections as we did before. Let ab be ten and zb be six, so the product of its square by az would be one hundred and forty-four, which equals the number and its side is bc. There is no doubt that bc must be greater than five, because the cube of five is one hundred and twenty-five, so the parallelepiped whose base is the square of zb and height za, equals the cube bc. Consequently, their bases are equivalent to their heights; I mean the ratio of the square of zb to the square of bc is the same as the ratio of bc to za. From z we draw a perpendicular which intersects the hyperbola at m and completes mb. The rectangle mb is equal to the rectangle ch because their sides are equivalent; I mean the ratio of zb to bc is the same as the ratio of bc to zm. Hence the ratio of the square of zb to the square of bc is the same as the ratio of zb to zm, and this ratio is the same as the ratio of bc to za. Thus the ratio of zb to zm is as the ratio of bc to za, and by interchanging, the ratio of zb to bc is the same as zm to za.

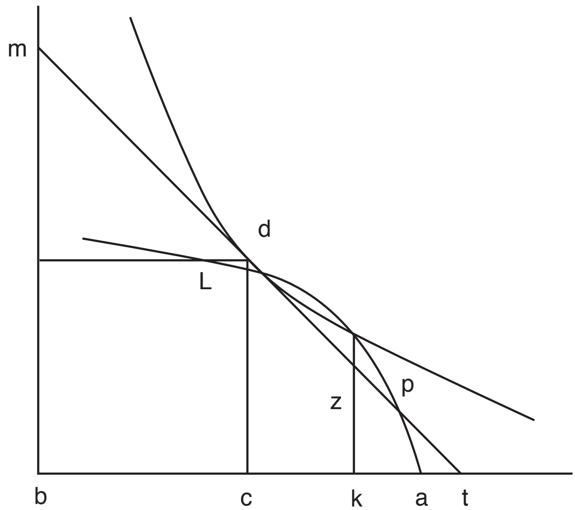
So the four lines are in sequence: zb, bc, zm, za. Thus the square of zm is equal to the product of bc and za; and bc is the right side of the parabola whose axis is ab and vertex a. It follows that zm is one of the lines of order.

So the point m is on the parabola. But it was on the hyperbola. Hence the two sections must meet. This shows that Abu-Aljood was wrong when he claimed that the sections would not meet. That is the required result.

To make it more clear, we let ab equal eighty and bc – which is the side of the cube that is equal to the number – equal forty-one, which is larger than ac. So the point d lies outside the parabola. Assume the parabola passes through l. Hence, the line lc is the root of one thousand five hundred and ninety-nine, which is slightly less than forty. We make tc equal bc and bm equal bt; then we connect th, so it is tangent to the parabola as we have shown. We take out ak to equal one fourth of ac and we draw a perpendicular to it that meets the conic section at p. So the ratio of the square of lc to the square of kp is the same as the ratio of ac to ak, because both are among the lines of order of the parabola.

Apollonius showed this in diagram yt of article a, so kp would be half lc which is slightly less than twenty, and ct forty-one, and ak nine and three-fourths and at equals two. So tk is eleven and three-fourths, because the ratio of kz to kt is the same as the ratio of mb to bt and they are equal, so the line kz equals kt, and hence the line zp is greater than eight, and it is inside the tangent line to the hyperbola, so in such a case, it must be within the hyperbola.

It is true that the two sections could not intersect if bc is larger than ca, but that is not true in all cases, and Abu-Aljood made a mistake in his reasoning.



If you want to find numerical examples, you could find some as follows: "add a (volume of) parallelepiped to a given line, which is less than it by a cube, but equal to another given parallelepiped".

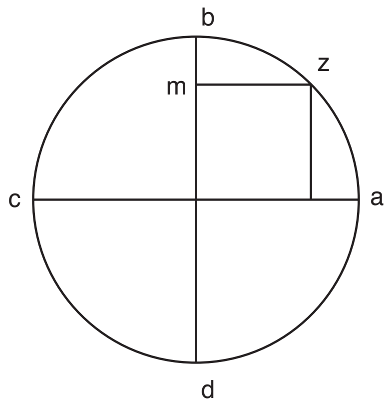
If the side of the cube that is equal to the given parallelepiped equals or is less than half of the line, then it can be solved. If it is greater than half of the line, there could be cases with no possible solution, as we have shown.

This is an Article of Abi-Al-Fath Omer Bin Ibraheem Al-Khayami

In The Name of God, Most Gracious Most Merciful From Him We Seek Exact Knowledge, And on Him We Depend

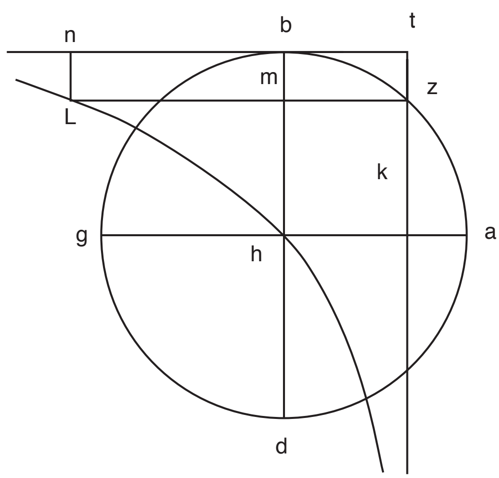
We want to divide the quarter ab (arc) of the circle abcd into two halves, at the point z, then draw the perpendicular zm to the diagonal bd so that the ratio of ah to zm is as the ratio of hm to mb, where h is the center of the circle and ah is its radius.

We draw a circle with center h, and draw a diagonal ac. Then we draw another diagonal bd so that the two diagonals are orthogonal. From a point z (on the arc ab) we draw the perpendicular zm to bd, such that the ratio of ah to zm is as the ratio of hm to mb. Then, we draw the two perpendiculars kzt and tbn, in a such a way that bn equals (in length) ah and we complete the rectangle tl.



Because the ratio of ah to zm is as the ratio of hm to mb, and bn equals ah, we get the ratio of bn to zm is the same as the ratio of hm to mb. Further, the product of bn and mb equals the product of zm and hm, as shown by Euclid in section yo of article o of the book of *Elements*. Further, the product of bn and mb equals the area of the rectangle bl, and the product of zm and hm equals the area of the rectangle mk, so the two rectangles mk and bl have the same area. We make the rectangle mt common so the rectangle th equals (in area) the rectangle tl.

Now, if we construct a hyperbola that does not meet the lines kt and tn but passes through the point h (as shown by Apollonius in figure nt of the first article of the book of conics, and in figures o and h of the second article of the same book), then this hyperbola must pass through the point l, as shown by reflecting figure eight of the second article of the book of sections.



The point h is of known position, and the line bn is of known position and magnitude. However, the point l is not of known position when the section is constructed, for if it were of known position, then the point m would be of known position because the line ml is of known magnitude. So the line bm would be of known magnitude, and consequently the whole figure would be totally known.

Also, the line tk is not of known position, for if it were, then the point t would be of known position. But if t were of known position, then the line tb would be of known magnitude. However, if tb were of known magnitude, then the whole figure would be totally known, which is not the case, since the goal is to determine the figure.

So if the point l were of known position, or the line tk were of known position, then the figure could be constructed and the conclusion would be reached easily. But knowing the location of either of them is not an easy task.

So if a researcher avoids this method and uses the book of sections, he will get the result in a different way.

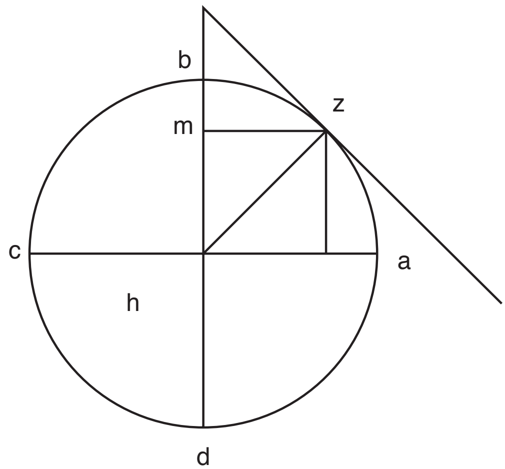
I mentioned this method, though it is difficult, as a preliminary introduction for the student. I did not complete and build it geometrically because of its difficulty and the lack of many basics and introductions on conics. Whoever wants to continue this method using conic sections can do so, once he understands the method that I will present. Though this method also lacks some introductions on conic sections, it is much easier than the above-mentioned one, and more fruitful.

I say with the help of God:

From the construction we did at the beginning, we found the ratio of ah to zm equals the ratio of hm to mb. From the point z we draw the line zt that is tangent to the circle, as Euclid explained in figure yo of section c of the book of *Elements*. We extend hb to intersect the tangent at t then we draw zh. Since the triangle hzt is a right-angled triangle (at z), and the perpendicular zm was drawn from z to the base of the triangle, it follows from figure m of section y (of the book of *Elements*) that the ratio of hm to mz is as the ratio of mz to mt. So the square of mz equals the product of hm and mt. But the square of mz equals the product of dm and mb. It follows that the product of dm and mb equals the product of hm and mt, and consequently, the ratio of md to hm is as the ratio of mt to mb, as shown in figure yo of section o (of the book of *Elements*). So the ratio of hc to hm is as the ratio of bt to mb. But the ratio of ah to zm is as the ratio of hm to mb and by transitivity, we have the ratio of ah to hm is as the ratio of zm to mb.

But it was shown that the ratio of ch to hm equals the ratio of bt to bm. Hence the ratio of zm to mb is as the ratio of bt to mb. Now, it is known (figure t of article h of the book of *Elements*) that quantities with the same ratio to a fixed quantity must be equal. Hence zm equals bt, and zh equals hb, and consequently hz plus zm equals ht.

The analysis leads to a right-angled triangle, with the condition that the hypotenuse equals one of the two sides of the right angle plus the perpendicular from the right angle to the hypotenuse. Every time we construct such a right-angled triangle, we are able to put together this figure in the right geometrical way. This introduction, I mean this triangle with this property, is of great benefit in figures of these types. It has many other properties; we will mention some so the researcher can use it in many similar problems.



I say: this triangle cannot be an equilateral triangle. For, if the side hz equals zt, then hm equals mt, and the perpendicular would be equal to each one of them, and ht would be equal to twice the perpendicular. Further, the sum of hz and the perpendicular, which we assumed to equal the hypotenuse, would be greater than the hypotenuse. This is a contradiction.

And I say: hz is smaller than zt, for if it were greater than it, then hm would be greater than mt. Further, mz (which is a middle line between the two lines hm and mt) would be greater than mt. But it was assumed that mz equals tb. So tb would be greater than tm, which is impossible, since tb is a piece of tm. Hence, the triangle as such, has the property that the side that is smaller than the perpendicular equals the one which is larger, and that is what we wanted to show.

Among other properties of such a triangle: The greater of the two sides of the right angle equals the sum of the smaller side plus the segment of the hypotenuse (formed by the perpendicular from the right angle to the hypotenuse) that meets the smaller side of the angle.

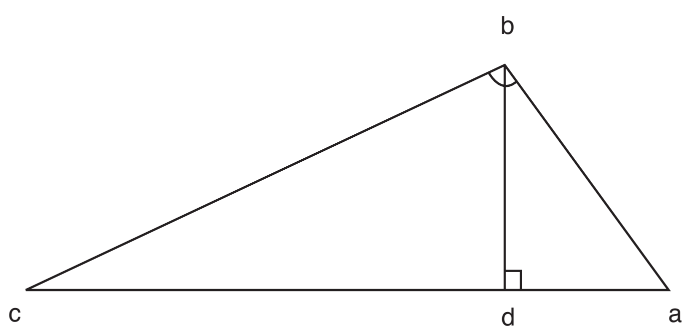
Our example will be based on the figure on page 48. I say: the sum (of the lengths) of hz and hm equals the side zt.

Proof: The ratio of hd to hm is as the ratio of tb to bm, so by composition, we get the ratio of dm to mh is as the ratio of tm to mb. Hence, the ratio of dm to mt is as the ratio hm to mb. But the ratio of hm to mb is as the ratio of hz to zm. And the ratio of hz to zm is as the ratio of zt to mt due to the similarity of the two triangles hzt and zmt. Hence the ratio of zt to mt is as the ratio of dm to mt, and this implies that zt equals md, which in turn equals the sum of hz and hm. So the sum of hz and hm equals zt, and that is what we wanted to show.

After what we have already introduced, we construct the triangle abc with right angle at b. A perpendicular bd is drawn from b to ac. We assume that the sum of ab and the perpendicular bd equals ac, so the analysis leads to a known situation. Then we compose things to obtain the triangle with the above-mentioned property.

To follow in the steps of the honorable ancients, who were experts in simplifying abstract methods by using algebraists' terminologies in these problems, we will use their terminology. These terminologies make multiplication and division much easier.

We draw ad to be a specified length, say of length ten. Put bd to be unknown, then multiply it by itself to make it square. Multiply the ten by itself to get one hundred, then sum to get a hundred and a square. This is the square of ab, as noted in figure mz of article a of the *Elements*. Further, the ratio of ac to ab is as the ratio of ab to ad because the two triangles abc and abd are similar. It follows that the product of ac by ad equals the square of ab. If we divide the square of ab (which equals one hundred and a square) by ad, which equals ten, the outcome will be ten plus one tenth of a square, which equals ac. But we have assumed that ac equals the sum of ab and bd. Hence the sum of ab and bd equals ten and one tenth of a square. Subtracting bd which is the unknown, from that, the result is ten plus one tenth of a square minus unknown (which is ab). Multiplying this by itself gives a hundred plus three squares plus one tenth of a tenth of square-square minus twenty of the unknown and minus one fifth of cube equals a hundred plus square ($100 + 3x^2 + \frac{1}{100}x^4 - 20x - \frac{1}{5}x^3 = 100 + x^2$).



Simplifying gives two squares plus one tenth of tenth of a square-square equals twenty of the unknown plus one fifth of cube. Divide by the unknown to get twice the unknown plus one tenth of tenth of cube equals twenty plus one fifth of square. Multiply (both sides of the equation) by a hundred to get: cube plus two hundreds of the unknown equals twenty squares plus two thousands. This analysis lead to an equation involving four objects of different ranks which cannot be solved using plane geometry of the cube, so conic sections will be needed.

Before we prove our claim using conics, we call the attention of the reader to an idea that will be an instigation to the reader of this article to acquire knowledge and master the part that we call attention to. Thanks to almighty God for his beneficences upon many of his worshippers. For, speaking of beneficences is great thanks to the giver, as God says in his book "speak of your God's beneficences". So the reader of this article should not think that this is showing off. Showing off is a practice of the arrogant, and they love it. Such people try to understand a little science. Once they have comprehended it (the little science), they think they know about all sciences, and God forbid that we allow ourselves to believe in such a way that prevents us from understanding the facts and winning salvation.

I say: what algebraists call square-square is an imaginary concept in continuous quantities. It has no existence in any way in materialistic objects. For continuous quantities, the terms square-square, square-cube and cube-cube are used to denote the number (coefficient) of the object (variable). Things share the sort of the quantity (all variables have the same type of coefficient: a real number), as is continuously shown by He who has the ultimate knowledge (God).

The things that algebraists use to denote objects and quantities are: number, root, square and cube. The number has to be taken as an abstract concept. It has no existence unless it is individuated by things. The root's status in continuous quantities is as the status of the straight line (in geometrical figures). The square's status is as that of a rectangle with equal sides and right angles whose side is what we call the root. The cube is the solid bounded by six equal squares whose sides are equal with right angles, any of its side is what we call the root and any of its faces is what is called the square, so the cube is the result of multiplying the root by itself, then multiplying the result by the root. This was shown and proved by Euclid in figure $\underline{\text{yz}}$ of article $\underline{\text{yc}}$ in the book of *Elements*.

Square-square, which, to the algebraists, is the product of the square by itself, has no meaning in continuous objects. This is because how can one multiply a square, which is a surface, by itself? Since the square is a two-dimensional object (geometrical figure), and two-dimensional by two-dimensional is a four-dimensional object. But solids cannot have more than three dimensions.

All objects in algebra are generated from these four genera. And anyone who says that algebra is a trick to determine unknown numbers is wrong. So don't pay attention to these people. It is true that algebra and equations are geometrical things, as proved in article $\underline{\text{b}}$ of the book of *Elements* in figures $\underline{\text{h}}$ and $\underline{\text{w}}$.

Now, whoever said: square-square plus three squares equals twenty-eight; he halved the squares then multiplied it by itself and then added the number; and took the root of the result to equal five and a half; then subtracted half the squares to get four which the square, and the square of the square is sixteen; and he thought that he deduced the square of the square using algebra: is very feeble in his thinking. This is because he did not deduce the square of the square but rather he deduced the square. It is exactly as if he said: square plus three roots equals twenty-eight, then he determined the root using the second reduction, and concluded that the square of this root is the square of the square, which is a secret from which you will come to know other secrets.

Let us go back to what we were occupied with:

We say: the first three genera (sorts), I mean numbers, roots and squares, when equated come down to six different equations (branches): three of which are single-term equations, and three are multi-term equations. The unknowns can be determined using the second article as was mentioned and explained in the books of algebraists. But if one studies equations involving cubes, then solids are needed especially cones and their sections, since the cube is a solid.

There are three single-term equations involving cubes: cube equals squares, which is equivalent to root equals numbers; and cube equals roots which is equivalent to square equals numbers, and cube equals numbers. There is no way to solve these equations except by using numerical methods intended to determine cubes; or else by using geometrical methods where a parallelepiped is constructed to equal a given parallelepiped. In such methods, conic sections are very much needed, and those who do not know conics need other techniques.

The multi-term equations involving cubes are of two types: either three-term equations or four-term equations. The three-term equations are:

(1) Cube plus squares equals numbers, which can be solved only by using conics.
(2) Cube plus squares equals roots, which can be treated as the equation: square plus roots equals numbers.
(3) Cube plus numbers equals roots, which can be solved only by using conics.
(4) Cube plus numbers equals squares, which can be solved only by using conics.
(5) Cube plus roots equals numbers, which can be solved only by using conics.
(6) Cube plus roots equals squares, which can be treated as square plus number equals root.
(7) Squares plus roots equals cube, which can be treated as roots plus numbers equals square.
(8) Squares plus numbers equals cube, which can be solved only by using conics.
(9) Roots plus numbers equals cube, which can be solved only by using conics.

So these are nine types of three-term equations, three of which can be deduced from the second book of *Elements*, and six of which can be solved only by using conics.

The four-term equations are:

(1) Cube equals squares plus roots plus numbers.

(2) Cube plus roots plus numbers equals squares.
(3) Cube plus squares plus numbers equals roots.
(4) Cube plus squares plus roots equals numbers.
(5) Cube plus square equals roots plus numbers.
(6) Cube plus roots equals squares plus numbers.
(7) Cube plus numbers equals squares plus roots.

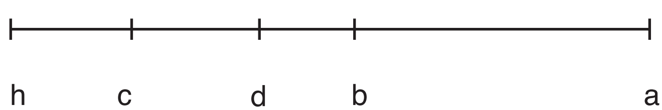
These are seven types of four-term equations, none of which can be solved except by using conics.

So, there are thirteen multi-term equations involving cubes, all of which cannot be solved except by using conics; and one single-term equation that can be solved only by using conics, which is: cube equals numbers.

Ancient mathematicians, who did not speak our language, either did not do any work in relation to this, or their work did not reach us and was not translated to our language.

As for modern mathematicians who speak our language, the first one who needed a type of three-term equation out of the fourteen types, was Al-Mahani, the geometer. He was trying to prove the premise, which was taken for granted by Archimedes, in figure d of article b in the book on "the sphere and the cylinder". Archimedes said: the two lines ab and bc are of known magnitude (length), joined along the same line. Further, the ratio of bc to ch, is known, so ch is known, as shown in *Data*. Then he said: we make the ratio of cd to ch equal to the ratio of the square of ab to ad.

But he did not say how we know this, because necessarily this needs conic sections, and he did not introduce in the book (his work) anything that needs conics except this, and so he took it for granted.



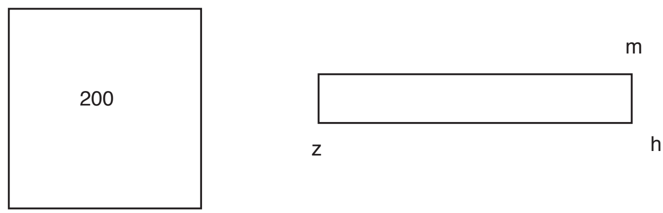
And the fourth proposition was to try to divide the sphere by a plane into two parts in a pre-given ratio. Mahani used algebraists' terminology to simplify things. But when analysis led him to equations involving numbers, squares and cubes, and he was not able to solve them by using conic sections, he concluded firmly that these equations have no solution. This gentleman, despite his deep understanding (and high achievement) of this industry (mathematics) could not solve certain of the previous types. Some time later, Abu Zafar Al-Khazin excelled in mathematics and discovered a new technique for solving these equations and explained it in an article (treatise). Also Abu Nasr Bin Iraq, a freedman of the commander of the faithful, from the city of Khawarizm, was trying to solve the premise that was taken for granted by Archimedes to deduce the length of the side of a heptagon inscribed in a circle, which is based on the square having the above-mentioned property. He was using the terminology of the algebraists, and the analysis led him to: cube plus squares equals numbers, which he solved using conic sections. This man was certainly among the high-ranking mathematicians.

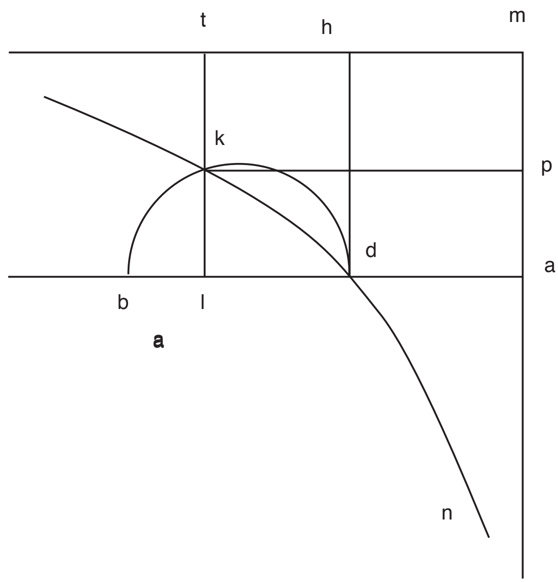
The problem that baffled Abu Sahel Al-Qohi, Aba Al-Wafaa Al-Bozjani, Aba Hamid Al-Saghani, and a group of their colleagues who devoted themselves to the Right Honorable "Adot Aldawla" (The governor) in the city of peace is: divide ten into two numbers so that the sum of their squares plus the outcome of dividing the greater by the smaller equals seventy-two. The analysis always led to squares equal cube plus roots plus numbers. These gentlemen were puzzled by the problem for a long time, till Abu-Aljood solved it and they kept the solution in the library of the Samanid kings. So these are three sorts: two of which are three-term equations, the third one is a four-term equation, while the fourth one is a single-term equation, I mean cube equals numbers, which has been solved by gentlemen who preceded us, but none of their work on the remaining ten sorts had reached us, nor any of their work on the above classification.

If I live for some time and success is my companion, I will write a self-contained article that includes the fourteen sorts with all of their types and branches, distinguishing the possible from the impossible ones – for several of the types require some conditions to be valid – and that contains many introductions that will be of great benefit to the principles of this industry, holding on the robe of success granted by God, depending on Him, for all seek help from him. He is the owner of power, praise be to His greatness.

Now, after these premises we go back to our problem, which is: determine a cube that satisfies the equation: the cube plus two hundreds of its side equals twenty squares of its side plus two thousands. To do that, we draw the line ab to equal the number of the squares (which is twenty). We draw another line, hz, to equal two hundreds, and the line hm to equal one, so the area of the rectangle mz equals two hundreds. Then we draw a square, with side am, to equal the rectangle mz, as was deduced from figure yd of article b (of the book of *Elements*). Let the side am be perpendicular to ab which equals the root of two hundreds. Further, ad equals ten, which is the result of dividing the number by the number of the roots; noting that the number is two thousands and the number of the roots is two hundreds. So dividing two thousands by two hundreds we get ten. Also db equals ten.

On db we construct the semicircle dkb, and we extend dh to be parallel to am, then we complete the rectangle ah. Now, we draw the hyperbola ndk that passes through d and does not meet am and mh, as was shown by the honorable Apollonius in proposition nt of the first article of the book of conics, and in proposition h and w in the second article of that book. This construction cannot be done without these three propositions, which is: the hyperbola ndk intersects the circle at k. From k we construct kl perpendicular to ab. I say: the side al is the side of the cube that satisfies: the cube plus two hundreds of its square equals twenty of the squares of al plus two thousands.





Proof: We extend lk until it intersects the line mh at t, then we draw kn parallel to al. Because kt is parallel to dh and kn is parallel to ad, the rectangle al equals the rectangle km. This is because the two points k and d are on a hyperbola that does not intersect the lines am and mt; and from each of the two points we draw two lines to the two other lines that do not meet the hyperbola and are parallel to their homologues (the other two lines from the other point). The honorable Apollonius proved this in figure c of article b of the book of conics. The circle dkb is of known location since its diagonal db is of known location and magnitude. The two lines am and mt are of known location, and the point n is of known location, so the conic ndk is of known location. Further, the point k and the line kl are of known locations. It follows that the point l is of known location. Also, the point a is of known location and consequently the line al is of known magnitude. All these deductions are clear in the book of *Data*.

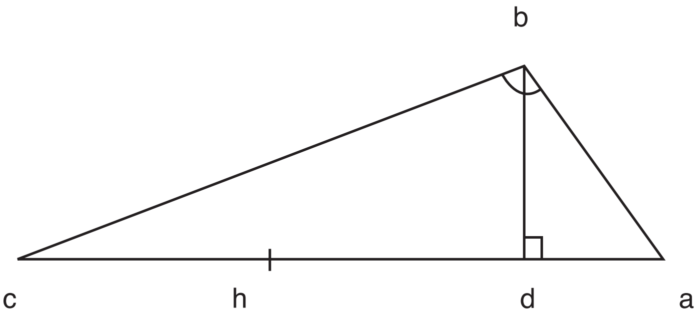
We have shown that the rectangle ah equals the rectangle km. Remove hp, which is common, to get the rectangle dp equals kh. We construct the rectangle dk common, so the rectangle ak equals the rectangle dt, which are of equal angles because their angles are right angles. Hence their sides are equivalent in ratio as was shown by Euclid in proposition yd of article o (of the book of *Elements*), so the ratio of al to lt is as the ratio of dl to lk, and hence their squares are proportional. Thus the ratio of the square of al to the square of lt is as the ratio of the square of dl to the square of lk. Also, the ratio of dl to lk is as the ratio of lk to lb. Consequently, the ratio of the square of dl to the square of lk is as the ratio of dl to lb. It follows that the ratio of the square of al to the square of lt is as the ratio of dl to lb. Hence the product of the square of al and the line lb equals the product of the square of lt and the line dl. Take the product of the square of lt and ad to be a common factor, so the product of the square of lt and al equals the product of the square of lt and ad and equals the product of the square of al and lb. But the square of lt equals the number of the sides, I mean two hundreds, and al is the side of the cube. So, two hundreds of the side of the cube equal the product of the square of lt and ad and equal the product of the square of al and lb. But as we have shown, the product of the square of lt and ad equals two thousands. It follows that two thousands plus the product of the square of al and lb equals two hundreds of the side of the cube.

We take the cube of al, which equals the product of the square of al by al, a common factor. So the cube of al plus two hundreds of the side of the cube equals two thousands plus the product of the square of al and al plus the product of the square of al and lb. But the product of al and al plus the product of the square of al and lb equals the product of the square of al and ab. However we have assumed ab to equal twenty. Hence the product of the square of al and ab equals twenty of the squares of al. So the cube of al plus two hundreds of (the length of) al equals two thousands plus twenty squares of the side of the cube. This is what we wanted to show.

Now, after what we have shown, we construct the triangle abc with ad (rational) equal to ten. Then db is the line al which we have proved is of known magnitude. I don't mean of known (magnitude) length, for there is a difference. I mean by saying of known magnitude what Euclid meant in his book of *Data*: a line to which an equal magnitude (of its length) can be found.

So putting things in order, we can assume ad equals ten. We draw bd perpendicular to ad and equal to the line al in the preceding figure. Then we join ab and draw from b the perpendicular bc. Extend ad to intersect the perpendicular from b at the point c. This triangle abc must be a right-angled triangle at b. So ab plus bd equals the hypotenuse ac, and ab plus ad equals bc, and that is what we wanted to show.

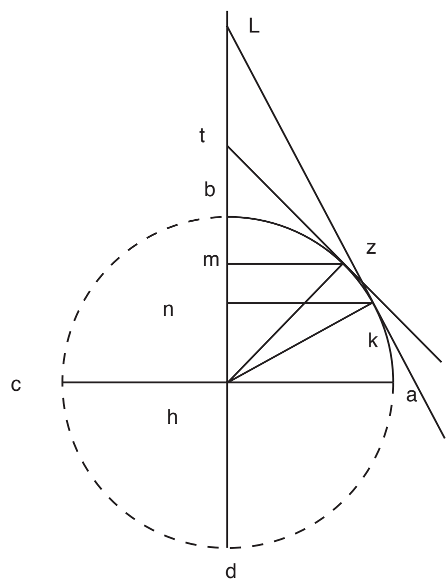
Consider the circle abcd with ab one quarter of its circumference. Draw the two diameters ac and bd to intersect at a right angle, and assume h is the center of the circle. Remove ch from the line cd of the triangle abc in the preceding figure, so that ch equals the perpendicular bd. Divide the radius of the circle (which equals hb in the preceding figure), at a point m, into two parts in such a way that the ratio between them is as the ratio of ad to dh of the triangle abc, as was shown by Euclid in proposition h of article w of his book, the *Elements*. We draw the perpendicular mz, and then we join hz. From z we draw the tangent zt to the circle, and we extend hb until it intersects zt at the point t. Hence the triangle hzt is similar to the triangle abc in the preceding figure.



Proof: The angle zhm equals the angle bac, for if not then one of them is larger than the other, say bac. Draw from h on the line hb the angle khl that is equal to bac. From k, we draw the tangent kl to the circle. This tangent intersects ht at l, so the triangle hkl is similar to the triangle abc since their angles are equal. We draw the perpendicular kn from k to hb so hk plus kn equals hl. Since hb equals hk, it follows that bl equals kn, and the ratio of ln to kn is as the ratio of cd to db. Consequently, the ratio of nl to lb is as the ratio of dc to ch. We conclude that the ratio of nb to bl is as the ratio of dh to hc, and the ratio of hc (which is equal to db) to da is as the ratio of bl (which is equal to kn) to nc. So from equal ratios we get the ratio of hn to nb is as the ratio of ad to dh. But we have made the ratio of hm to mb equal to the ratio of ad to dh, so the ratio of hn to nb is as the ratio of hm to mb. However, hn (the first) is smaller than hm (the third). It follows necessarily that nb (the second) is smaller than mb (the fourth), based on proposition yd in chapter five of the book of *Elements*. But it was assumed that it is larger. This is impossible. So the angle zhm is not smaller than bac in the triangle abc, neither could it be larger than it; so the triangle zhm is similar to abc. Consequently hz plus zm equals ht. It follows that bt equals zm, and the product of dm and mb equals the square of mz. Also, the product of hm and mt equals the square of mz. Hence the product of dm and mb equals the product of hm and mt. So the four lines are proportional, as shown in proposition yo of article d (of the book of *Elements*), and the ratio of dm (the first) to mh (the second) is as the ratio of mt (the third) to mb (the fourth). In conclusion: the ratio of hd to hm is as the ratio of bt to bm, and dh equals ah and bt equals zm. So the ratio of ah to hm is as the ratio of zm to mb. By interchanging ratios we get the ratio of ah to zm is as the ratio of hm to mb.

Hence we divided a quarter of the circumference of a circle into two pieces at a point z, from which we drew the perpendicular zm so that the ratio of ah (which is the radius) to zm equals the ratio of hm to mb, and that is what we wanted to show.

Whoever wants to deduce the above result using arithmetic will not be able to do so, since results that are deduced using conic sections cannot be deduced using arithmetic. But if he is satisfied by guessing (and approximating), then he must go back to the tables of hypotenuses in the book of Al-Magest, or the tables of sines and arrows of a reliable ephemeris. He must look for an arc (of the circle) where the ratio of sixty (which is the radius of the circle by assumption) to its sine (the central angle with this arc) is as the ratio of the cosine to its arrow. So one finds this arc to be very close to 57 (nz) degrees of the 360 (shs) degrees of the circle, its sine is close to 50 (n) degrees, its arrow is close to 27 (kz) parts and one third of a part, and its cosine is close to thirty-two degrees and two thirds of a degree. One can go on refining these estimates until no difference can be noticed between the exact and the approximate values.



This is what was possible to discuss and present in that direction, despite our thoughts being scattered, and despite the many worries of life that hinder us from tackling these particular cases. And had it not been for the nobility of the council (may his nobility be for ever) and the right of the querist (may God support him always), I would not bother with these small details, but rather occupy myself with what is more important. We thank God in all cases, and in God we trust to guide us to good things in life.

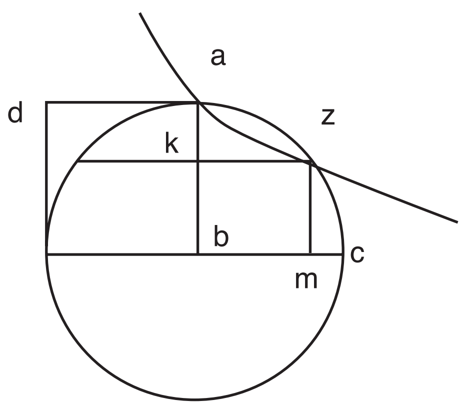
The article is complete, and peace be upon the Seal of the Prophecy.

Problem

Given a quarter of a circle ac with assumed center b, and we want to divide ab into two parts, as you already know.

We construct on ab a square (equal sides and angles), say ah, so that the lines bh and hd are of known location, and the point a is of known location. At a we draw the hyperbola az that does not meet the lines bh and hd, so this hyperbola is of known location. We join ac, and necessarily it is tangent to the hyperbola and it is inside the circle. Consequently, the hyperbola must intersect the circle, say at z, and z will be of known location. We draw the two perpendiculars zm and zk. I say the work is done.

Proof: The two points a and z are on the hyperbola. From each of these two points, two lines were drawn to meet the two lines that do not meet the hyperbola and parallel to the two lines from the other point. So the rectangle zh equals the rectangle ah. We remove kh, which is common to both. What is left is km, which equals kd, and the corresponding angles of the two rectangles are equal, so their sides are equivalent: the ratio of ad to kz is as the ratio of bk to ka.



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CENTER FOR MUSLIM CONTRIBUTION TO CIVILIZATION

The Center for Muslim Contribution to Civilization, a non-government, non-profit making cultural organization, strives to lead Muslims and non-Muslims alike to a better understanding of the Muslim contribution to civilization and to a better knowledge of Islam.

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