The background of the book cover is a composite image. The left half is a solid dark purple. The right half features a grayscale photograph of a microscope's eyepiece and a portion of a ruler with numerical markings. The text is overlaid on these elements.

**Fourth Edition**

# Practical Physics

**G.L. SQUIRES**

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## Practical Physics

*Practical Physics* demonstrates the purposive and critical approach that should be made to all experimental work in physics. It does not describe a systematic course of experiments, but is intended as a companion to any undergraduate course of practical work.

The text is in three parts. The first deals with the statistical treatment of data, the second with experimental methods, and the third with such essential matters as keeping efficient records, accuracy in calculations, and writing good, scientific English. The text is liberally illustrated with examples and exercises, with solutions to the latter.

The new edition includes a treatment of the  $\chi^2$  distribution, a section on atomic clocks and their applications, worked examples based on spreadsheets, and additional exercises. The examples and references have been brought up to date.

Although intended mainly for undergraduates, *Practical Physics* has proved of interest to senior school students, teachers, and research workers, not only in physics, but also in other branches of science. Previous editions have been translated into German, Dutch, Spanish, Russian, Polish, and Arabic.

GORDON SQUIRES has been a Lecturer in Physics at the University of Cambridge and a Fellow of Trinity College, Cambridge since 1956. His field of research is thermal neutron scattering and he has published many papers in leading scientific journals. He is also the author of *Introduction to the Theory of Thermal Neutron Scattering* (Dover Publications) and *Problems in Quantum Mechanics* (Cambridge University Press) as well as an article on quantum mechanics in the *Encyclopaedia Britannica*. Since retiring as a University Lecturer in 1991, Dr Squires has been the Curator of the Cavendish Laboratory Museum.

# Practical Physics

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and Fellow of Trinity College, Cambridge*

FOURTH EDITION



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## PREFACE TO THE FOURTH EDITION

The present edition retains the basic outlook of the book, namely to demonstrate the purposive and critical approach which should be made to all experimental work in physics. But I have made a number of changes and additions in response to new experimental methods and the widespread use of computers, which I hope will add to its usefulness.

Substantial changes have been made in chapter 7, in which a selection of techniques are analysed to show the art and craft of the experimenter. I have added a section on the measurement of time and frequency, which includes an account of the caesium atomic clock and the present universal time scale. This is followed by a description of the Global Positioning System, which, based on atomic clocks, enables position on the surface of the Earth to be determined to very high precision. Timing techniques have a number of elegant and ingenious features, which, combined with the importance of their practical applications, make them instructive and interesting, both at the elementary and advanced level.

I have added an appendix on the  $\chi^2$  distribution. The goodness of fit test based on this distribution finds widespread application in the physical, biological, medical, and social sciences. Many students have had an introduction to the topic at school and have learnt to apply the formulae of the test, but I feel that valuable insight is gained from seeing a derivation of the distribution, which I have given in a not too formal manner.

Spreadsheets are now in common use, and I have included some worked examples based on them. I have taken the opportunity to add to the exercises, and have brought some of the examples, references, and definitions of units up to date.

I would like to thank Mr J. Acton, Dr C. Bergemann, Professor M. F. Collins, and Dr D. Kennedy for helpful discussions and for comments on the revised parts of the book, and Mr A. Squires both for useful comments and for providing Fig. E.2b.

G. L. SQUIRES

*July 2000*

## PREFACE TO THE FIRST EDITION

Experimental physics has occupied some of the finest intellects in the history of man, but the fascination of the subject is not always apparent in an undergraduate course of practical work. This book is about experimental physics and it is intended for undergraduates, but it does not describe a systematic course of experiments, nor is it a handbook of experimental techniques. Instead, it sets out to demonstrate a certain outlook or approach to experimental work. It is intended as a companion to a general course of practical work. My aim is to make the student more critical of what he does and more aware of what can be done, and in this way to make the course more interesting and meaningful.

The book is in three parts. The first is on the statistical treatment of data. I have tried to give the statistical theory not as an exercise in mathematics but rather as a tool for experimental work. This is perhaps the most difficult part of the book, and the student should not worry if he does not grasp all the mathematical details at first. He should read through the chapters to get a general understanding – and then go ahead and use the results. He can always return and master the proofs at a later stage. The second part is on experimental methods. I discuss a selection of instruments, methods, and experiments with a view to showing the craft of the experimenter. The selection is arbitrary – though I have tried to illustrate the points with methods that are useful in themselves. The third part concerns such matters as keeping efficient records, getting arithmetic right, and writing good scientific English.

The examples have been kept fairly simple. Apart from the account of the measurement of the magnetic moment of the electron, the level of the material is roughly that of a first-year undergraduate course. But I think that a wider range of students – from intelligent sixth-formers to research students – could benefit from the experimental 'awareness' that the book is trying to foster.

The experiment to measure the magnetic moment of the electron is an advanced one and contains several ideas beyond those of an average first-year course. I have tried to give sufficient explanation to make it intelligible to someone in his second or third year who has had an introduction to quantum mechanics. The experiment is a rewarding one to study, but the whole account may be omitted at first reading without detriment to the understanding of the rest of the book.

I would like to thank Professor O. R. Frisch, Professor R. G. Chambers, Mr E. S. Shire, Dr J. Ashmead, Dr J. R. Waldram, Dr B. D. Josephson, and Dr

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G. L. SQUIRES  
*September 1967*



# 1 The object of practical physics

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This book is intended to help you to do practical physics at college or university: its aim is to make the laboratory work more useful and profitable. We may start by asking what is the object of laboratory work in a university physics course. There are several possible objects. Laboratory work may serve

- (a) to demonstrate *theoretical ideas* in physics,
- (b) to provide a familiarity with *apparatus*,
- (c) to provide a training in *how to do experiments*.

Let us consider each of these in turn.

Seeing something demonstrated in practice is often a great help in understanding it. For example, interference in light is not an intuitive concept. The idea that two beams of light can cancel each other and give darkness takes a little swallowing, and most people find it helpful to be given a visual demonstration. A demonstration is useful for another reason – it gives an idea of orders of magnitude. The interference fringes are in general close together, which indicates that the wavelength of light is small compared with everyday objects. But the demonstration is no substitute for a proper explanation, which goes into the details of geometry and phase relations. So the first object, the demonstration of theoretical ideas, has a definite but limited usefulness.

The second object is perhaps more important, but it is necessary to say exactly what is meant by 'apparatus' in this context. In any practical course you will handle a number of instruments, such as oscilloscopes, timers, thermometers, and so on, and the experience you gain from using them should prove useful. However, if you eventually do scientific work of some kind, the range of instruments you could conceivably work with is enormous. No practical course could possibly teach you to use them all. What the course should do is to train you to use instruments *in general*. There is a certain attitude of mind that an experimenter should adopt when handling any instrument, and this the course should try to instil. But this is part of the third object which is the most important of all.

The phrase 'how to do experiments' may sound vague, so let us try to be more specific. The primary object – or set of objects – of practical physics is to train you to

- (a) plan an experiment whose precision is appropriate to its purpose,
- (b) be aware of and take steps to eliminate systematic errors in methods and instruments,
- (c) analyse the results in order to draw correct conclusions,
- (d) estimate the precision of the final result,
- (e) record the measurements and calculations accurately, clearly, and concisely.

All this adds up to saying that the main object of a course in practical physics is to train you to be a competent experimenter. But the course can do still more. It can show the way physics works.

Physics is one of the natural sciences, that is to say, it is part of our attempt to understand the natural world. When we are confronted by a situation in the natural world, the way we proceed in physics is to select what we think are the essential features. For example, the Greeks saw that a moving body came to rest and they therefore said that a force is necessary to keep a body moving. Galileo and Newton observed the same phenomenon, but they said that the coming to rest of the body is not an essential feature of the situation. It depends on friction; in the absence of friction a body keeps moving. If we try to do an experiment to test this view, we find that we cannot completely eliminate friction or other retarding forces, but we can make such forces small, and the smaller we make them the farther the body goes before coming to rest. So it is reasonable to believe that in the limiting case of zero friction the motion will remain unchanged as stated in Newton's first law.

This is the way physics works. We select what we think are the essential features in an actual physical situation. From them we make a generalization, or theory, and from the theory, deductions. We test a deduction by doing an experiment. But the deduction refers to an idealized or theoretically simple situation. In order to test it we have to create this simple situation in the messy, complicated, natural world, which is often a difficult thing to do.

In lectures you are taught the theory of the subject. The physical world is described in terms of the features which the current theories say are essential. These features tend to be the only ones you hear about, and you may well come to feel that they constitute the entire world, instead of a specially selected part of it. Moreover, everything fits together so smoothly that you can easily lose sight of the genius and effort that went into creating the subject. The most effective antidote to this is going into the laboratory and seeing the complications of real life.

By doing practical physics, then, you learn at first hand some of the obstacles to testing a theory, to measuring what you want to measure and not something else, and you learn how to overcome them. But above all you get an insight into physics as a whole, into the way experiment and theory interact, which is the very essence of the subject.

## Part 1

### Statistical treatment of data





### 2.1 The importance of estimating errors

When we measure a physical quantity, we do not expect the value obtained to be exactly equal to the true value. It is important to give some indication of how close the result is likely to be to the true value, that is to say, some indication of the precision or reliability of the measurements. We do this by including with the result an estimate of its error. For example, we might measure the focal length  $f$  of a lens and give the final result as

$$f = (256 \pm 2) \text{ mm.} \quad (2.1)$$

By this we mean that we expect the focal length to be somewhere in the range 254 to 258 mm. Equation (2.1) is really a probability statement. It means, not that we are *certain* that the value lies between the limits quoted, but that our measurements indicate that there is a certain *probability* of its doing so. In chapter 3 we shall make this statement more precise.

Estimates of errors are important because without them we cannot draw significant conclusions from the experimental results. Suppose, for example, we wish to find out whether temperature has an effect on the resistance of a coil of wire. The measured values of the resistance are

200.025  $\Omega$  at 10 °C

200.034  $\Omega$  at 20 °C.

Is the difference between these two values significant? Without knowing the errors we cannot say. If, for example, the error in each value of the resistance is 0.001  $\Omega$ , the difference is significant, whereas if the error is 0.010  $\Omega$ , then it is not.

Once the result of an experiment has been obtained it goes out into the world and becomes public property. Different people may make use of it in different ways. Some may use it in calculations for a practical end; others may want to compare it with a theoretical prediction. For example, an electrical engineer may want to know the resistivity of copper in order to design a transformer, while a solid state physicist may want to know the same quantity to test a theory of electrons in metals. Whatever use people make of an experimental result, they will want to know whether it is sufficiently precise for their purpose. If they have drawn some conclusions from the result, they will want to know how much

confidence to place in them. To answer such questions, an estimate of the error in the result is necessary, and it is the responsibility of the experimenter to provide it.

Now although the experimenter may not be able to foresee all the possible uses of his results, he should be aware of some of them. No experiment should be done in a vacuum – at least not in an intellectual one. If the experiment is being done to test a theory, the experimenter should have some idea how precise the result needs to be in order to provide a useful comparison with the theoretical prediction. So the idea of using error estimates to draw conclusions from the results of an experiment applies also in reverse. That is to say, the purpose of the experiment often determines the error that can be tolerated, and this in turn may have an important influence on the experimental procedure.

It might be thought that every experiment should be done as precisely as possible, but that is an unrealistic point of view. Life is finite, so are the experimenter's resources, and so also, unless he is a genius, is his capacity for taking pains. Therefore it is important to plan and carry out the experiment so that the precision of the final answer is appropriate to the ultimate object of the experiment. Suppose, in the example given at the beginning of this section, that we are only interested in the resistance of the coil because we want to use it as a standard resistance in the temperature range  $10^{\circ}\text{C}$  to  $20^{\circ}\text{C}$ , and that the precision required is 1 part in 10 000. A measurement of the resistance with an error of  $0.010\ \Omega$  would then be quite adequate, and to strive to reduce the error to  $0.001\ \Omega$  would be a waste of time. On the other hand, to measure the resistance to only  $0.05\ \Omega$  would be even worse because the measurements would be useless for their purpose.

Just as the final result of an experiment should be obtained to an appropriate degree of precision, so also should the values of the various measured quantities within the experiment. Few experiments are so simple that the final quantity is measured directly. We usually have to measure several primary quantities and bring the results together in order to obtain the quantity required. The errors in the primary quantities determine that in the final result. In general the primary errors contribute different amounts to the final error, and the latter is minimized if the finite resources of time, apparatus, and patience available are concentrated on reducing those errors that contribute most to the final error.

So we see that the idea of errors is not something of secondary or peripheral interest in an experiment. On the contrary, it is related to the purpose of the experiment, the method of doing it, and the significance of the results.

## 2.2 Systematic and random errors

Errors may be divided into two kinds, systematic and random. A *systematic* error is one which is constant throughout a set of readings.\* A *random* error is one

\* This definition is actually too restrictive – some systematic errors are not constant. But in order to give the basic ideas we restrict the discussion here to the simple case. More general cases are considered in chapter 8.

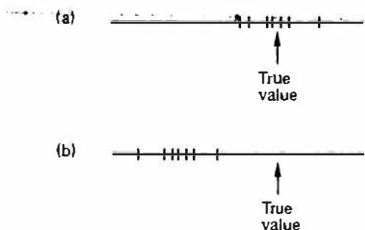


Fig. 2.1. Set of measurements (a) with random errors only and (b) with systematic plus random errors. Each point indicates the result of a measurement.

which varies and which is equally likely to be positive or negative.

Random errors are always present in an experiment and, in the absence of systematic errors, they cause successive readings to spread about the true value of the quantity – Fig. 2.1a. If in addition a systematic error is present, the readings spread, not about the true value, but about some displaced value – Fig. 2.1b.

Suppose that the period of a pendulum is measured by means of a stopclock, and the measurement is repeated many times. Errors in starting and stopping the clock, in estimating the scale divisions, small irregularities in the motion of the pendulum, all these cause variations in the results of successive measurements and may be regarded as random errors. If no other errors are present, some results will be too high and others too low. But if, in addition, the clock is running slow, all the results will be too low. This is a systematic error.

It should be noticed that systematic and random errors are *defined* according to whether they produce a systematic or random effect. So we cannot say that a certain source of error is inherently systematic or random. Returning to the example, suppose that every time we measure the period we use a different clock. Some clocks may be running fast and others slow. But such inaccuracies now produce a random error.

Again, some sources of error may give rise to both systematic and random effects. For example, in operating the clock we might not only start and stop it in a slightly irregular manner in relation to the motion of the pendulum, thus producing a random error, but we might also have a tendency to start it too late and stop it too early, which would give rise to a systematic error.

It is convenient to make a distinction between the words *accurate* and *precise* in the context of errors. Thus a result is said to be *accurate* if it is relatively free from systematic error, and *precise* if the random error is small.

### 2.3 Systematic errors

Systematic errors often arise because the experimental arrangement is different from that assumed in the theory, and the correction factor which takes account of this difference is ignored. It is easy to give examples of effects that may lead to systematic error. Thermal emfs in a resistance bridge, the resistance of the leads in a platinum thermometer, the effect of the exposed stem in a mercury thermometer, heat losses in a calorimetric experiment, counting losses due to the dead-time in a particle counter are but a few. Another common source of systematic error is the one mentioned earlier – inaccurate apparatus.

Random errors may be detected by repeating the measurements. Furthermore, by taking more and more readings we obtain from the arithmetic mean a value which approaches more and more closely to the true value. Neither of these points is true for a systematic error. Repeated measurements with the same apparatus neither reveal nor do they eliminate a systematic error. For this reason systematic errors are potentially more dangerous than random errors. If large random errors are present in an experiment, they will manifest themselves in a large value of the final quoted error. Thus everyone is aware of the imprecision of the result, and no harm is done – except possibly to the ego of the experimenter when no one takes notice of his or her results. However, the concealed presence of a systematic error may lead to an apparently reliable result, given with a small estimated error, which is in fact seriously wrong.

A classic example of this was provided by Millikan's oil-drop experiment to measure  $e$ , the elementary charge. In this experiment it is necessary to know the viscosity of air. The value used by Millikan was too low, and as a result the value he obtained for  $e$  was

$$e = (1.591 \pm 0.002) \times 10^{-19} \text{ C.}$$

This may be compared with the present value (Mohr and Taylor 2000)

$$e = (1.602\,176\,46 \pm 0.000\,000\,06) \times 10^{-19} \text{ C.}$$

Up till 1930, the values of several other atomic constants, such as the Planck constant and the Avogadro constant, were based on Millikan's value for  $e$  and were consequently in error by more than  $\frac{1}{2}\%$ .

Random errors may be estimated by statistical methods, which are discussed in the next two chapters. Systematic errors do not lend themselves to any clear-cut treatment. Your safest course is to regard them as effects to be discovered and eliminated. There is no general rule for doing this. It is a case of thinking about the particular method of doing an experiment and of always being suspicious of the apparatus. We shall try to point out common sources of systematic error in this book, but in this matter there is no substitute for experience.

## 3 Treatment of a single variable

### 3.1 Introduction

Suppose we make a set of measurements, free from systematic error, of the same quantity. The individual values  $x_1, x_2$ , etc., vary owing to random errors, and the mean value  $\bar{x}$  (i.e. the arithmetic average) is taken as the best value of the quantity. But, unless we are lucky,  $\bar{x}$  will not be equal to the true value of the quantity, which we denote by  $X$ . The question we are going to consider is how close we expect  $\bar{x}$  to be to  $X$ . Of course we do not know the *actual* error in  $\bar{x}$ . If we did, we would correct  $\bar{x}$  by the required amount and get the right value  $X$ . The most we can do is to say that there is a certain probability that  $X$  lies within a certain range centred on  $\bar{x}$ . The problem then is to calculate this range for some specified probability.

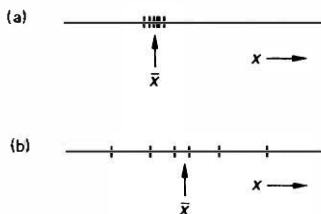


Fig. 3.1. Results of successive measurements of the same quantity. The mean  $\bar{x}$  is expected to be closer to the true value for set (a) than for set (b).

A clue to how we should proceed is provided by the results shown in Fig. 3.1. On the whole, for the results in Fig. 3.1a we would expect  $X$  to be fairly close to  $\bar{x}$ ; whereas for those in Fig. 3.1b we would not be greatly surprised if there were quite a large difference. In other words, the larger the spread in the results, the larger we would expect the error in  $\bar{x}$  to be. The whole of the present chapter is concerned with putting this qualitative statement on a firm quantitative basis. We assume throughout that no systematic errors are present.

Table 3.1. *Measurements of the resistance  $R$  of a coil*

$R/\Omega$	$R/\Omega$
4.615	4.613
4.638	4.623
4.597	4.659
4.634	4.623

### 3.2 Set of measurements

Denote the values of  $n$  successive measurements of the same quantity by

$$x_1, x_2, \dots, x_n. \quad (3.1)$$

The number  $n$  is not necessarily large and in a typical experiment might be in the range 5 to 10. The mean is

$$\bar{x} = \frac{1}{n} \sum x_i. \quad (3.2)$$

(Whenever the symbol  $\sum$  appears in the present chapter, the summation is to be taken from  $i = 1$  to  $i = n$ .)

To fix our ideas let us consider a specific experiment in which the resistance of a coil is measured on a bridge. The measurement is made  $n = 8$  times. The results are listed in Table 3.1. The mean of these values is  $4.625 \Omega$ . We require a quantity that gives a measure of the spread in the 8 values, from which we shall estimate the error in the mean. To define such a quantity we need to introduce the idea of a *distribution* – one of the basic concepts in the theory of statistics.

### 3.3 Distribution of measurements

**(a) Introduction.** Although we have only  $n$  actual measurements, we *imagine* that we go on making the measurements so that we end up with a very large number  $N$ . We may suppose  $N$  is say 10000 000. (Since we are not actually making the measurements, expense is no object.) We call this hypothetical set of a very large number of readings a *distribution*. The basic idea to which we shall constantly return is that *our actual set of  $n$  measurements is a random sample taken from the distribution of  $N$  measurements*.

We may represent any set of measurements by a *histogram*. This is true whether it be a set of a small number  $n$  of measurements or a distribution of a large number  $N$ . To construct a histogram we simply divide the range of measured values into a set of equal intervals and count how many times a measurement occurs in each interval. The width of the intervals is arbitrary and is chosen for convenience. Figure 3.2 shows a histogram for the measurements in Table 3.1.

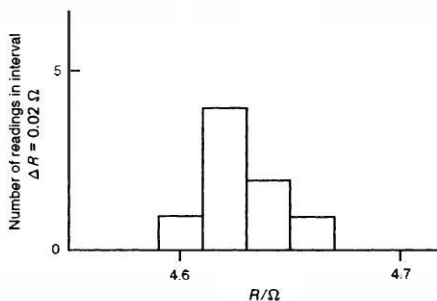


Fig. 3.2. Histogram of the readings given in Table 3.1.

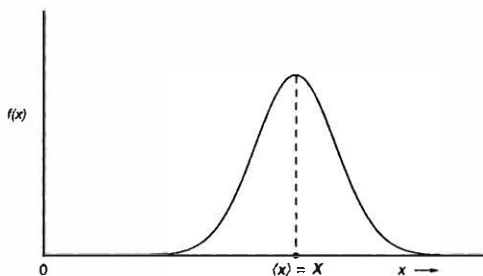


Fig. 3.3. Typical function for distribution of measurements.

This histogram has a jagged appearance because it represents only a few values. However, suppose we represent a distribution in this way. The number of measurements  $N$  is so large that we may make the width of the intervals very small and – provided the measuring instrument gives sufficiently fine readings – still have an appreciable number of readings in each interval. If we plot, instead of the histogram itself, the *fraction* of the  $N$  readings in each interval as a function of the value of the measurement, we shall get a smooth curve. We may then define a function  $f(x)$ , known as the *distribution function*, whose significance is that  $f(x) dx$  is the fraction of the  $N$  readings that lie in the interval  $x$  to  $x + dx$ . In other words,  $f(x) dx$  is the *probability* that a single measurement taken at random from the distribution will lie in the interval  $x$  to  $x + dx$ . We shall not specify the exact form of  $f(x)$  at this stage but we expect a typical distribution function to look something like that shown in Fig. 3.3.

From its definition  $f(x)$  satisfies the relation

$$\int_{-\infty}^{\infty} f(x) dx = 1. \quad (3.3)$$

Notice the infinite limits in the integral. We do not expect any measurements with values greatly different from the true value  $X$  in an actual experiment. In particular, many quantities are of a nature such that negative values are impossible. Therefore any function  $f(x)$  that we use to represent a distribution must become very small as the difference between  $x$  and  $X$  becomes large. For such functions no difficulty arises from the infinite limits, and they are taken for mathematical convenience.

We shall use the symbol  $\langle \rangle$  to denote an average over all the measurements in the distribution. An important average is the *mean* of the distribution

$$\langle x \rangle = \int_{-\infty}^{\infty} x f(x) dx. \quad (3.4)$$

Since the number of measurements in the distribution is very large, and they are free from systematic error,  $\langle x \rangle$  may be taken as equal to the true value  $X$ .

**(b) Standard error in a single observation.** The error in a measurement with value  $x$  is

$$e = x - X. \quad (3.5)$$

The rms (root-mean-square) value of  $e$  for all the measurements in the distribution is denoted by  $\sigma$  and is known as the *standard deviation of the distribution*.\* Thus  $\sigma$  is defined by the equation

$$\sigma^2 = \langle e^2 \rangle = \int_{-\infty}^{\infty} (x - X)^2 f(x) dx. \quad (3.6)$$

The standard deviation is a measure of the spread of the distribution, i.e. of the scatter of the measurements. A distribution representing a precise set of measurements will be highly peaked near  $x = X$  and will have a small value for  $\sigma$ ; while one representing an imprecise set will have a large scatter about  $X$  and a large value for  $\sigma$  (Fig. 3.4). We take the quantity  $\sigma$  as our measure of the error in a single observation, and it is therefore also referred to as the *standard error in a single observation*.

**(c) Standard error in the mean.** We now proceed to define a quantity that specifies the error in the mean of a set of  $n$  measurements.

Let us go back to the set of 8 measurements given in Table 3.1. We have said that they are to be regarded as a random sample taken from the distribution of single measurements. Imagine the distribution to be represented by a bin containing  $N$  balls, each of which is labelled with the value of a single measurement. The set of measurements in Table 3.1 may be regarded as the result of

\* The quantity  $\sigma^2$  is known as the *variance* of the distribution.



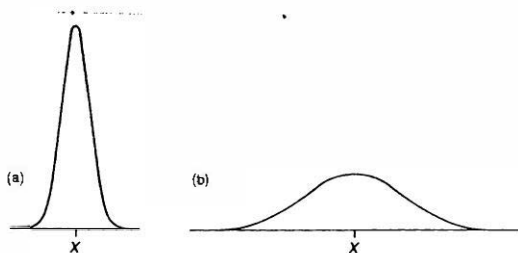


Fig. 3.4. Distribution function  $f(x)$  for (a) a precise set of measurements (small value of  $\sigma$ ), and (b) an imprecise set of measurements (large value of  $\sigma$ ). Note that the area under the two curves is the same because both functions satisfy the relation (3.3).

drawing 8 balls at random from the bin. Suppose now we have a second bin, initially empty, and a supply of blank balls. We look at our 8 measurements, calculate their mean, write the answer on a blank ball and put it in the second bin. We put the set of 8 single-measurement balls back into the first bin, stir the contents, and take out, at random, a second set of 8 balls. We again calculate the mean of the 8 values, write it on a blank ball, and put it in the second bin. We continue the process a large number of times, always drawing the same number, 8, of balls from the first bin. We end up with a large number of values in the second bin, which represents another distribution, namely a distribution of means of 8 measurements.

We denote the standard deviation of this distribution by  $\sigma_m$ . We take it as our measure of the error in the mean of a single set of  $n$  measurements, and it is therefore known as the *standard error in the mean*.

To sum up,  $\sigma$  is the standard deviation of the distribution of single measurements, and  $\sigma_m$  is the standard deviation of the distribution of the means of sets of measurements, each set containing the same number  $n$  of single measurements.  $\sigma$  represents the error in a single measurement, and  $\sigma_m$  represents the error in the mean of  $n$  measurements.

**(d) Relation between  $\sigma$  and  $\sigma_m$ .** There is a simple relation between  $\sigma$  and  $\sigma_m$  which we now prove. Consider one set of  $n$  measurements  $x_1, \dots, x_n$ . The error in the  $i$ th reading is

$$e_i = x_i - X, \quad (3.7)$$

where  $X$  is the true value of the quantity, which is of course unknown. The error in the mean is

$$E = \bar{x} - X = \left( \frac{1}{n} \sum x_i \right) - X = \frac{1}{n} \sum (x_i - X) = \frac{1}{n} \sum e_i. \quad (3.8)$$

Therefore

$$E^2 = \frac{1}{n^2} \sum e_i^2 + \frac{1}{n^2} \sum_{i \neq j} e_i e_j. \quad (3.9)$$

This is for a single set of  $n$  measurements. Now imagine that, as before, we take a large number of sets, each set consisting of the same number  $n$  of single measurements. Each set will have its own set of values for  $e_1, \dots, e_n$ , and its own value of  $E$ . Equation (3.9) will be true for each set. We add the equations for all the sets and then divide by the number of sets, that is to say, we average (3.9) over all the sets. The average of  $\sum e_i^2$  is  $n\langle e^2 \rangle$ . The average of each term in the double sum is zero, because the errors  $e_i$  and  $e_j$  are independent, and the average of each is zero. We therefore arrive at the result

$$\langle E^2 \rangle = \frac{1}{n} \langle e^2 \rangle. \quad (3.10)$$

By definition

$$\sigma_m^2 = \langle E^2 \rangle \quad \text{and} \quad \sigma^2 = \langle e^2 \rangle. \quad (3.11)$$

Equation (3.10) thus becomes

$$\sigma_m = \frac{\sigma}{\sqrt{n}}, \quad (3.12)$$

i.e. the standard error in the mean of  $n$  observations is  $1/\sqrt{n}$  times the standard error in a single observation.

The value of  $\sigma$  depends only on the precision of the individual measurements and is independent of their number; whereas the value of  $\sigma_m$  can be reduced by increasing  $n$ . However, since  $\sigma_m$  decreases only as  $1/\sqrt{n}$ , it becomes more and more unprofitable to take readings of the same quantity. Rather we should try to reduce  $\sigma_m$  by reducing  $\sigma$ , i.e. by taking a more precise set of readings in the first place.\*

### 3.4 Estimation of $\sigma$ and $\sigma_m$

(a) *Standard method.* We have defined the quantities  $\sigma$  and  $\sigma_m$  that we are going to take as our measures of the error in a single measurement and in the mean. It remains to show how we can calculate, or more correctly how we can best estimate, them from the actual measurements. We need to estimate only one of them, because we can then use (3.12) to obtain the other.

The best estimate of  $\sigma$  is  $[(1/n) \sum e_i^2]^{1/2}$ , but the errors  $e_i$  come from the true

\* A splendid example of this approach is to be found in the experiment to measure  $g$  precisely – see section 7.3 (d).

value  $X$  and hence are not known. A way round the difficulty is provided by working in terms of *residuals*.

The residual  $d_i$  of the  $i$ th measurement is defined by

$$d_i = x_i - \bar{x}. \quad (3.13)$$

Unlike the error, the residual is a known quantity. We denote the rms value of the  $n$  residuals by  $s$ , i.e.

$$s^2 = \frac{1}{n} \sum d_i^2. \quad (3.14)$$

The quantity  $s$  is called the *standard deviation of the sample*. From (3.7) and (3.8)

$$x_i - \bar{x} = e_i - E. \quad (3.15)$$

Therefore

$$\begin{aligned} s^2 &= \frac{1}{n} \sum (x_i - \bar{x})^2 = \frac{1}{n} \sum (e_i - E)^2 \\ &= \frac{1}{n} \sum e_i^2 - 2E \frac{1}{n} \sum e_i + E^2 \\ &= \frac{1}{n} \sum e_i^2 - E^2. \end{aligned} \quad (3.16)$$

This is for one set of  $n$  measurements. As before, we take the average of this equation over a large number of sets in the distribution and obtain the result

$$\langle s^2 \rangle = \sigma^2 - \sigma_m^2. \quad (3.17)$$

From (3.12) and (3.17) we have

$$\sigma^2 = \frac{n}{n-1} \langle s^2 \rangle, \quad (3.18)$$

and

$$\sigma_m^2 = \frac{1}{n-1} \langle s^2 \rangle. \quad (3.19)$$

The quantity  $\langle s^2 \rangle$  is not known. Our best estimate of it is  $s^2$ , obtained by evaluating (3.14). Substituting this value and taking the square root gives

$$\sigma \approx \left( \frac{n}{n-1} \right)^{\frac{1}{2}} s, \quad (3.20)$$

$$\sigma_m \approx \left( \frac{1}{n-1} \right)^{\frac{1}{2}} s. \quad (3.21)$$

We thus have estimates of  $\sigma$  and  $\sigma_m$  in terms of quantities that are known.\*

\* The symbol  $\approx$  signifies that (3.20) and (3.21) are not strictly equations. The values of the right-hand sides depend on the particular set of  $n$  measurements and are not in general exactly equal to  $\sigma$  and  $\sigma_m$  – see section 3.7.

Table 3.2. Estimation of  $\sigma$  and  $\sigma_m$  for the measurements in Table 3.1

Resistance $R/\Omega$	Residual $d/\text{m}\Omega$	$(d/\text{m}\Omega)^2$
4.615	-10	100
4.638	13	169
4.597	-28	784
4.634	9	81
4.613	-12	144
4.623	-2	4
4.659	34	1156
4.623	-2	4
mean = 4.625		sum = 2442

(b) *Worked example.* As an example we show how  $\sigma$  and  $\sigma_m$  are estimated for the set of measurements in Table 3.1, which are listed again in the first column of Table 3.2. The first step is to calculate the mean, which is  $4.625\ \Omega$ . From the mean we calculate the residual of each measurement. For example, the residual of the first measurement is

$$d_1 = (4.615 - 4.625)\Omega = -10\ \text{m}\Omega. \quad (3.22)$$

The residuals and their squares are listed in the second and third columns of Table 3.2. Then

$$s^2 = \frac{1}{n} \sum d_i^2 = \frac{2442}{8} \times 10^{-6} \Omega^2, \quad s = 0.017\ \Omega, \quad (3.23)$$

$$\sigma \approx \left[ \frac{n}{n-1} \right]^{\frac{1}{2}} s = \left( \frac{8}{7} \right)^{\frac{1}{2}} \times 0.017 = 0.019\ \Omega, \quad (3.24)$$

$$\sigma_m = \frac{\sigma}{\sqrt{n}} \approx \frac{0.019}{\sqrt{8}} = 0.007\ \Omega. \quad (3.25)$$

The result of a set of measurements is quoted as  $\bar{x} \pm \sigma_m$ . So in the present case our best estimate of the resistance of the coil is

$$R = 4.625 \pm 0.007\ \Omega. \quad (3.26)$$

(c) *Programmed calculator.* Calculators, programmed to calculate  $\sigma$ , use the standard method, but they do not evaluate  $s$  from (3.14) because that requires the value of  $\bar{x}$ , which is not known until all the numbers are fed in. However, there is an alternative expression for  $s$  that avoids the difficulty. From (3.2), (3.13), and (3.14) we have

$$\begin{aligned}
 s^2 &= \frac{1}{n} \sum (x_i - \bar{x})^2 \\
 &= \frac{1}{n} \left[ \sum x_i^2 - 2\bar{x} \sum x_i + n\bar{x}^2 \right] \\
 &= \frac{1}{n} \sum x_i^2 - \left( \frac{1}{n} \sum x_i \right)^2.
 \end{aligned} \tag{3.27}$$

Combining this with (3.20) gives

$$\sigma^2 \approx \frac{n}{n-1} \left[ \frac{1}{n} \sum x_i^2 - \left( \frac{1}{n} \sum x_i \right)^2 \right] \tag{3.28}$$

$$= \frac{1}{n-1} \left[ \sum x_i^2 - \frac{1}{n} (\sum x_i)^2 \right]. \tag{3.29}$$

This is the expression used by a calculator programmed to evaluate standard deviations. As the numbers  $x_i$  are fed in, the calculator accumulates the values of  $\sum x_i^2$  and  $\sum x_i$ . It then uses (3.2) and (3.29) to evaluate  $\bar{x}$  and  $\sigma$ .

We may rewrite (3.28) as

$$\sigma^2 \approx \frac{n}{n-1} \left[ \overline{x^2} - (\bar{x})^2 \right], \tag{3.30}$$

where a bar over a quantity denotes the average over the set.  $\sigma$  is usually small compared to  $\bar{x}$ , so the difference between  $\overline{x^2}$  and  $(\bar{x})^2$  is usually small compared to the quantities themselves. For the measurements in Table 3.2

$$\overline{x^2} = 21.393\,24\,\Omega^2, \quad \text{and} \quad (\bar{x})^2 = 21.392\,94\,\Omega^2. \tag{3.31}$$

Since these quantities are almost equal, and we require the difference between them, they have to be evaluated very precisely to give an adequate estimate of  $\sigma$ . So the expression (3.29) is suitable only for a calculator or a computer. For a hand calculation the method of section (b) should be used.

**(d) Deviations from a general value of  $x$ .** Before leaving this section we prove one further result. Suppose that instead of taking the deviations of the readings from the mean  $\bar{x}$ , as we have done so far, we take the deviations from a different value of  $x$ . Denote by  $S(x)$  the rms deviation of the readings taken from  $x$ , so that

$$[S(x)]^2 = \frac{1}{n} \sum (x_i - x)^2. \tag{3.32}$$

Combining this equation with (3.27) we have

$$\begin{aligned}
 [S(x)]^2 - s^2 &= \frac{1}{n} \sum \left[ (x_i - x)^2 - (x_i - \bar{x})^2 \right] \\
 &= \frac{1}{n} \sum (x^2 - 2x_i x + 2x_i \bar{x} - \bar{x}^2) \\
 &= x^2 - 2\bar{x}x + 2\bar{x}^2 - \bar{x}^2 = (x - \bar{x})^2,
 \end{aligned}$$

i.e.

$$[S(x)]^2 = s^2 + (x - \bar{x})^2. \quad (3.33)$$

This demonstrates an important result. For a given set of readings the sum of the squares of the deviations is a minimum when the deviations are taken from the mean of the set. That is the reason why  $s^2$  is not an unbiased estimate of  $\sigma^2$ . It is slightly too small, as (3.18) shows.

### 3.5 The Gaussian distribution

We have not yet specified the exact form of the distribution function  $f(x)$ . The results derived so far are therefore independent of the distribution. However, to make further progress we need a specific function, and the one we shall take is

$$f(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma} \exp\left[-(x - X)^2/2\sigma^2\right]. \quad (3.34)$$

This function, specified by the two constants  $X$  and  $\sigma$ , is known as a *Gaussian*, and the distribution as a *Gaussian* or *normal* distribution. Its shape is shown in Fig. 3.5.

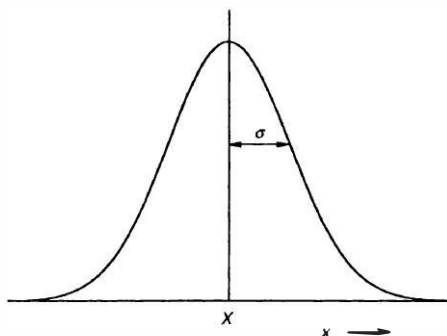


Fig. 3.5. The Gaussian distribution function. The points of inflexion are at  $x = X \pm \sigma$ .

Later in the chapter we mention the reasons for choosing the Gaussian, but at this stage we merely note that the function

- (i) is symmetric about  $X$ ,
- (ii) has its maximum value at  $X$ ,
- (iii) tends rapidly to zero as  $|x - X|$  becomes large compared with  $\sigma$ .

Clearly these are reasonable properties for a function representing a distribution of measurements containing only random errors.

Table 3.3. *Some useful integrals for the Gaussian distribution*

$$\begin{aligned}\int_{-\infty}^{\infty} \exp(-x^2/2\sigma^2) dx &= \sqrt{(2\pi)}\sigma \\ \int_{-\infty}^{\infty} x^2 \exp(-x^2/2\sigma^2) dx &= \sqrt{(2\pi)}\sigma^3 \\ \int_{-\infty}^{\infty} x^4 \exp(-x^2/2\sigma^2) dx &= 3\sqrt{(2\pi)}\sigma^5\end{aligned}$$

We show below that the constant  $\sigma$  in (3.34) is in fact the standard deviation of the function – hence the choice of the symbol. The purpose of the multiplying factor

$$\frac{1}{\sqrt{(2\pi)}} \frac{1}{\sigma}$$

is to make  $f(x)$  satisfy (3.3). We can see that it does so from the value of the first integral in Table 3.3. (The results in the table are proved in Appendix A.) Put  $X=0$ , which does not affect the argument. Then

$$\int_{-\infty}^{\infty} f(x) dx = \frac{1}{\sqrt{(2\pi)}} \frac{1}{\sigma} \int_{-\infty}^{\infty} \exp(-x^2/2\sigma^2) dx = 1. \quad (3.35)$$

The standard deviation of the function in (3.34) is obtained from the second integral in Table 3.3. We continue with  $X=0$ . By definition

$$\begin{aligned}(\text{standard deviation})^2 &= \int_{-\infty}^{\infty} x^2 f(x) dx \\ &= \frac{1}{\sqrt{(2\pi)}} \frac{1}{\sigma} \int_{-\infty}^{\infty} x^2 \exp(-x^2/2\sigma^2) dx \\ &= \sigma^2.\end{aligned} \quad (3.36)$$

It is readily verified that the points of inflexion of the function  $\exp(-x^2/2\sigma^2)$  occur at  $x = \pm\sigma$ , a convenient result for relating the standard deviation to the shape of a Gaussian.

### 3.6 The integral function

Suppose we have a symmetric distribution represented by a function  $f(x)$  for which  $X=0$ . We can ask what fraction  $\phi(x)$  of the measurements lie within the range  $-x$  to  $x$ . Since  $f(x) dx$  is by definition the fraction of readings between  $x$  and  $x + dx$ ,  $\phi(x)$  is given by

$$\phi(x) = \int_{-x}^x f(y) dy. \quad (3.37)$$

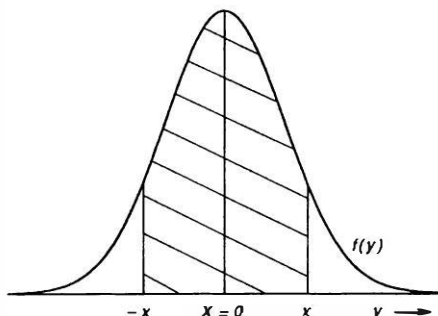


Fig. 3.6.  $\phi(x)$ , the fraction of measurements within  $\pm x$ , is the ratio of the shaded area to the total area under the distribution function  $f(y)$ .

We call  $\phi(x)$  the *integral function* of the distribution. It is equal to the shaded area in Fig. 3.6 divided by the total area under the curve.

For a Gaussian distribution with standard deviation  $\sigma$

$$\begin{aligned}\phi(x) &= \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma} \int_{-x}^x \exp(-y^2/2\sigma^2) dy \\ &= \sqrt{\left(\frac{2}{\pi}\right)} \frac{1}{\sigma} \int_0^x \exp(-y^2/2\sigma^2) dy.\end{aligned}\quad (3.38)$$

The last line follows because the function  $\exp(-y^2/2\sigma^2)$  is even, i.e. it has the same value for  $-y$  as for  $+y$ . The function  $\phi(x)$  depends on  $\sigma$ . It is convenient to have one table of values that can be used for all values of  $\sigma$ . The variable is therefore changed to  $t = y/\sigma$ . Put  $z = x/\sigma$ . Then

$$\phi(z) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^z \exp(-t^2/2) dt. \quad (3.39)$$

The function  $\phi(z)$  must be evaluated by numerical methods. It is tabulated in Appendix H.1 and plotted in Fig. 3.7.

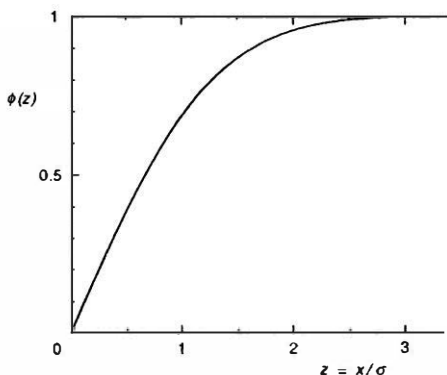
A few selected values of  $\phi(z)$  are given in Table 3.4. The numbers in the third column are worth remembering. We see that about two out of three observations lie within  $\pm\sigma$ . About 1 in 20 of the observations lie outside  $2\sigma$ , and about 1 in 400 outside  $3\sigma$ .

These results provide a quantitative basis for the statement that  $\sigma$  is a measure of the spread of the observations. They also provide a check that  $\sigma$  has been estimated correctly. For a set of readings with mean  $\bar{x}$ , roughly two-thirds of the readings should lie in the range  $\bar{x} \pm \sigma$ . We can also apply the results to the interpretation of  $\sigma_m$ , remembering that  $\sigma_m$  is the standard deviation of the



Table 3.4. *Values of the Gaussian integral function*

$z = x/\sigma$	$\phi(z)$	Approximate fraction of readings outside $z$ value
0	0	1 out of 1
1	0.683	3
2	0.954	20
3	0.9973	400
4	0.99994	16 000

Fig. 3.7. The integral function  $\phi(z)$  for the Gaussian distribution.

distribution of means of which  $\bar{x}$  is a member. Thus, when we quote the result of the measurements as  $\bar{x} \pm \sigma_n$ , the implication is that, in the absence of systematic error, the probability that the true value of the quantity lies in the quoted range is roughly two-thirds.

In addition to  $\sigma$ , another quantity sometimes used to specify the error in the measurements is the so-called *probable error*. It is defined as the value of  $x$  such that one-half of the readings lie within  $x$  of the true value. For the Gaussian distribution the probable error is equal to  $0.67\sigma$ . There is little point in having two similar measures of the error, and it is obviously convenient to settle on one of them. Though its significance is easy to appreciate, the probable error is not a very fundamental quantity, and the error commonly quoted nowadays is the standard error. We use it throughout the present book and only mention the probable error because it may be encountered in older books and papers.

### 3.7 The error in the error

It was stated in section 3.4 that the best estimate of  $\langle s^2 \rangle$  is provided by  $s^2$ . However,  $s^2$  is just the value we happen to get from the particular set of  $n$  readings. It is of interest to know how  $s^2$  varies from one set to another. The error in  $s^2$  is

$$u = s^2 - \langle s^2 \rangle. \quad (3.40)$$

In Appendix B we show that for a Gaussian distribution the mean value of  $u^2$  (taken from a large number of sets of  $n$  readings) is  $[2/(n-1)]\langle s^2 \rangle^2$ . Hence the fractional standard deviation of  $s^2$  is  $[2/(n-1)]^{1/2}$ , and provided  $n$  is fairly large, the fractional standard deviation of  $s$  is approximately half this value, i.e.  $[1/(2n-2)]^{1/2}$ .

The quantity  $1/(2n-2)^{1/2}$  is plotted against  $n$  in Fig. 3.8. The result provides a salutary warning against elaborate calculations of errors. Notice for example that for  $n=9$ , a not insignificant number of readings, the error estimate is only good to 1 part in 4.

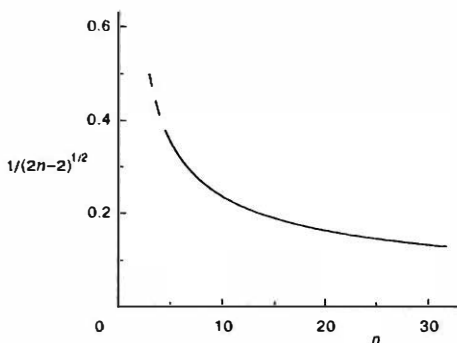


Fig. 3.8.  $1/(2n-2)^{1/2}$ , representing the fractional standard deviation of  $s$ , plotted against  $n$ , the number of measurements.

### 3.8 Discussion of the Gaussian distribution

So far we have used the Gaussian function, simply noting that its algebraic form is suitable for a distribution of measurements with only random errors. A theoretical justification for the function is provided by a theorem in statistics, known as the *central limit theorem*, which shows that, if each observation is the result of a large number of independent errors, the distribution tends to a Gaussian.

The assumption of a Gaussian distribution is related to taking the mean of a set of readings as the best value of the measured quantity. The word 'best' in this context is defined as follows. Suppose the distribution function has the form  $f(x - X)$ , where as usual  $X$  is the true value of the quantity. Let  $\varepsilon$  be the smallest quantity to which the measuring instrument is sensitive. (We suppose  $\varepsilon$  to be small – its actual value does not affect the argument.) The probability that we shall obtain the values  $x_1, x_2, \dots, x_n$  when we make  $n$  measurements is

$$f(x_1 - X) f(x_2 - X) \cdots f(x_n - X) \varepsilon^n. \quad (3.41)$$

The best value of  $X$  is defined to be the one which when inserted into (3.41) makes the quantity a maximum, i.e. it is the value which maximizes the probability of getting the particular set of measured values. Now it can be proved that if  $f(x - X)$  is a Gaussian, the best value of  $X$  is the mean of  $x_1$  to  $x_n$ , and conversely that if the best value of  $X$  is the mean, then the distribution function is a Gaussian (Whittaker and Robinson 1944, p. 218).

The Gaussian is the only distribution we shall use, but this should not be taken to imply that all experimental distributions in physics are of this form. Phenomena in which a random process gives rise to discrete measured values – for example, the counting of particles in atomic and nuclear physics – follow the Poisson distribution. This is discussed in Appendix D.

The results we have derived using the Gaussian distribution are in fact rather insensitive to the precise form of the distribution. This applies to the results in sections 3.6 and 3.7. We have seen – section 3.7 – that, quite apart from the question of the form of the distribution, the values obtained for the errors are, in the majority of cases, fairly crude estimates. Their uncertainties completely swamp effects due to the distribution being slightly non-Gaussian.

The main thing is to have some distribution which is (a) reasonable and (b) easy to handle algebraically. In most cases the Gaussian distribution fulfils both conditions very nicely. So, unless the measurements provide clear evidence to the contrary, we assume the distribution is Gaussian and use the formulae based on it. The one common instance of non-Gaussian distribution is when the measurements are discrete, being readings of an instrument to the nearest scale division, or of a digital instrument with a small variation of the last digit. This situation is discussed in chapter 5.

**Summary of symbols, nomenclature, and important formulae****A. Set of measurements***Quantities that are known*

measured values	$x_1, x_2, \dots, x_n$
mean	$\bar{x} = \frac{1}{n} \sum x_i$
residual for $i$ th reading	$d_i = x_i - \bar{x}$
standard deviation of sample	$s = \left( \frac{1}{n} \sum d_i^2 \right)^{\frac{1}{2}}$

*Quantities that are not known*

true value	$X$
error in $i$ th reading	$e_i = x_i - X$
error in mean	$E = \bar{x} - X$

**B. Distributions***Distribution of single measurements*

standard error	$\sigma = \langle e^2 \rangle^{\frac{1}{2}}$
----------------	--

*Distribution of means of  $n$  measurements*

standard error	$\sigma_m = \langle E^2 \rangle^{\frac{1}{2}}$
----------------	--

$\langle \rangle$  denotes the average over the distribution.

**C. Important relations**

$$\sigma_m = \frac{\sigma}{\sqrt{n}}$$

$$\sigma^2 = \frac{n}{n-1} \langle s^2 \rangle$$

$$\sigma_m^2 = \frac{1}{n-1} \langle s^2 \rangle$$

**D. Formulae for estimating  $\sigma$  and  $\sigma_m$** 

$$\sigma \approx \left[ \frac{\sum d_i^2}{n-1} \right]^{\frac{1}{2}} = \left[ \frac{\sum x_i^2 - \frac{1}{n} (\sum x_i)^2}{n-1} \right]^{\frac{1}{2}}$$

$$\sigma_m \approx \left[ \frac{\sum d_i^2}{n(n-1)} \right]^{\frac{1}{2}} = \left[ \frac{\sum x_i^2 - \frac{1}{n} (\sum x_i)^2}{n(n-1)} \right]^{\frac{1}{2}}$$

**E. Gaussian distribution**

$$f(x) = \frac{1}{\sqrt{(2\pi)}} \frac{1}{\sigma} \exp \left[ - (x - X)^2 / 2\sigma^2 \right].$$

Put  $X = 0$ . The fraction of readings between  $x$  and  $x + dx$  is  $f(z) dz$ , where

$$f(z) = \frac{1}{\sqrt{(2\pi)}} \exp(-z^2/2), \quad z = \frac{x}{\sigma}.$$

The fraction of readings between  $-x$  and  $+x$  is

$$\phi(z) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^z \exp(-t^2/2) dt.$$

$f(z)$  and  $\phi(z)$  are tabulated in Appendix H.I.

## Exercises

- 3.1 A group of students measure  $g$ , the acceleration due to gravity, with a compound pendulum and obtain the following values in units of  $\text{m s}^{-2}$ :

9.81, 9.79, 9.84, 9.81, 9.75, 9.79, 9.83.

Set out the values as in Table 3.2, and calculate the mean and the residuals. Hence estimate  $\sigma$ . Give the best estimate of  $g$ , together with its error, for the group.

- 3.2 In an undergraduate practical class in the Cavendish Laboratory there was an experiment, originally devised by Searle, to measure the Young modulus  $E$  for steel by applying a known load to a rod and measuring the deflection by an optical method based on Newton's rings. Although ingenious and capable of considerable precision in the hands of a skilled experimenter, such as Searle himself, the results obtained by the students were found to have a considerable scatter. The experiment was therefore replaced by one in which a horizontal steel beam was supported near its ends, and the deflection when a known load was applied at the centre was measured directly by a dial indicator.

The values obtained for  $E$  by the last 10 students who did the Newton's rings experiment and by the first 10 who did the dial indicator experiment are given below. The values are in units of  $10^{11} \text{ N m}^{-2}$ .

Newton's rings experiment 1.90, 2.28, 1.74, 2.27, 1.67, 2.01, 1.60, 2.18, 2.18, 2.00.

Dial indicator experiment 2.01, 2.05, 2.03, 2.07, 2.04, 2.02, 2.09, 2.09, 2.04, 2.03.

For each set of values, calculate the mean value of  $E$ , and estimate the standard error in the mean. Do the results indicate any systematic difference in the two experimental methods?

- 3.3 The thermal conductivity of copper at  $0^\circ\text{C}$  is

$$k = 385.0 \text{ W m}^{-1} \text{ K}^{-1}.$$

A large number of measurements of  $k$ , free from systematic error, form a Gaussian distribution with standard error

$$\sigma = 15.0 \text{ W m}^{-1} \text{ K}^{-1}.$$

What is the probability that a single measurement lies in the range

- (a) 385.0 to 385.1, (b) 400.0 to 400.1, (c) 415.0 to 415.1, (d) 370.0 to 400.0, (e) 355.0 to 415.0, (f) 340.0 to 430.0  $\text{W m}^{-1} \text{ K}^{-1}$ ?

- 3.4 In section 3.3 (d) we showed that the standard deviation  $\sigma_m$  of the distribution of the means of a sample of  $n$  measurements, and the standard deviation  $\sigma$  of the distribution of single measurements, are related by

$$\sigma_m = \frac{\sigma}{\sqrt{n}}.$$

This result was proved before we specified the form of the distribution function and is therefore true for *any* distribution function. To reinforce the point, consider a distribution which consists of only two values, namely,  $-1$  with probability 0.9 and  $+9$  with probability 0.1. Calculate the value of  $\sigma$ , and the value of  $\sigma_m$  for  $n=3$ , and show that  $\sigma_m = \sigma/\sqrt{3}$  even for this (highly non-Gaussian) distribution.

## 4.1 The treatment of functions

In most experiments we do not measure the final quantity  $Z$  directly. Instead we measure certain primary quantities  $A$ ,  $B$ ,  $C$ , etc. and calculate  $Z$ , which must be a known function of the primary quantities. For example, we might measure the density  $\rho$  of the material of a rectangular block by measuring the mass  $M$  and the dimensions  $l_x$ ,  $l_y$ ,  $l_z$  of the block. The functional relation between the quantity we require,  $\rho$ , and the primary quantities  $M$ ,  $l_x$ ,  $l_y$ ,  $l_z$  is

$$\rho = \frac{M}{l_x l_y l_z}. \quad (4.1)$$

Suppose that each primary quantity has been measured several times. Then, in the case of  $A$ , we have a best value  $\bar{A}$ , the mean of the measured values, and an estimate of its standard error  $\Delta A$ . (The latter is the  $\sigma_m$  of the previous chapter.) Similarly we have  $\bar{B}$  and an estimate of  $\Delta B$ . We assume that the measurements of the primary quantities are independent and therefore that the errors in them are also independent. By this we mean, for example, that if the value of  $\bar{A}$  happens to be high, the value of  $\bar{B}$  still has an equal probability of being high or low. From the values  $A = \bar{A}$ ,  $B = \bar{B}$ , etc., the best value of  $Z$  may be calculated. The problem we are going to consider is how to calculate the standard error  $\Delta Z$  in  $Z$  from the standard errors  $\Delta A$ ,  $\Delta B$ , etc. Although we are restricting the discussion to independent measurements, there are occasional situations where the assumption is not valid. The way  $\Delta Z$  is calculated in such cases depends on the way the primary errors are related; no general rule can be given. Exercise 4.2 provides an example of related errors.

(a) *Functions of one variable.* We consider first the case where  $Z$  is a function of only one variable  $A$ , for example

$$Z = A^2 \quad \text{or} \quad Z = \ln A.$$

We write this in general as

$$Z = Z(A). \quad (42)$$

(The symbol  $A$  is used both for the name of the primary quantity and for its value.)

If the true value of the primary quantity is  $A_0$ , the true value of  $Z$  is

$$Z_0 = Z(A_0) \quad (4.3)$$

—see Fig. 4.1. The error in a given value  $A$  is

$$E_A = A - A_0, \quad (4.4)$$

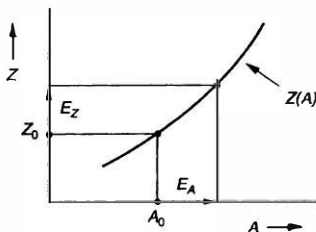


Fig. 4.1. Error  $E_Z$  in  $Z$  due to error  $E_A$  in  $A$ .

and this gives rise to an error  $E_Z$  in  $Z$ , where

$$E_Z = Z(A_0 + E_A) - Z(A_0) \quad (4.5)$$

$$\approx \frac{dZ}{dA} E_A. \quad (4.6)$$

The derivative  $dZ/dA$  is evaluated at  $A = A_0$ . The approximation in (4.6) is equivalent to the assumption that the error in  $A$  is sufficiently small for  $Z(A)$  to be represented by a straight line over the range of the measured values of  $A$ . The error in  $Z$  is therefore proportional to the error in  $A$ , the constant of proportionality being

$$c_A = \left( \frac{dZ}{dA} \right)_{A=A_0}. \quad (4.7)$$

We now allow  $A$  to vary according to the distribution of which  $\bar{A}$  is a member and take the root-mean-square of (4.6). This gives the result

$$\Delta Z = c_A \Delta A. \quad (4.8)$$

An important special case is  $Z = A^n$ , for which  $c_A = nA^{n-1}$ . Then

$$\frac{\Delta Z}{Z} = n \frac{\Delta A}{A}, \quad (4.9)$$

i.e., the fractional standard error in  $Z$  is  $n$  times that in  $A$ . We have already used this result (p. 22) for the case  $n = \frac{1}{2}$ .

**(b) Functions of several variables.** We next consider the case where  $Z$  is a function of two variables  $A$  and  $B$ ,



Table 4.1. *Combination of errors*

Relation between $Z$ and $A, B$	Relation between standard errors	
$Z = A + B$ $Z = A - B$	$(\Delta Z)^2 = (\Delta A)^2 + (\Delta B)^2$	(i)
$Z = AB$ $Z = A/B$	$\left(\frac{\Delta Z}{Z}\right)^2 = \left(\frac{\Delta A}{A}\right)^2 + \left(\frac{\Delta B}{B}\right)^2$	(ii)
$Z = A^n$	$\frac{\Delta Z}{Z} = n \frac{\Delta A}{A}$	(iii)
$Z = \ln A$	$\Delta Z = \frac{\Delta A}{A}$	(iv)
$Z = \exp A$	$\frac{\Delta Z}{Z} = \Delta A$	(v)

$$Z = Z(A, B). \quad (4.10)$$

The errors in  $A$  and  $B$  are

$$E_A = A - A_0, \quad E_B = B - B_0, \quad (4.11)$$

where  $A_0$  and  $B_0$  are the true values of  $A$  and  $B$ . As before we assume that  $Z$  is approximately a linear function of  $A$  and  $B$  in the range over which the measured values vary. Then the error in  $Z$  is

$$E_Z = c_A E_A + c_B E_B, \quad (4.12)$$

where the coefficients  $c_A$  and  $c_B$  are given by

$$c_A = \left(\frac{\partial Z}{\partial A}\right), \quad c_B = \left(\frac{\partial Z}{\partial B}\right). \quad (4.13)$$

The partial derivatives are evaluated at  $A = A_0, B = B_0$ .

From (4.12)

$$E_Z^2 = c_A^2 E_A^2 + c_B^2 E_B^2 + 2c_A c_B E_A E_B. \quad (4.14)$$

We take the average of this equation for pairs of values of  $A$  and  $B$  taken from their respective distributions. Since  $A$  and  $B$  are assumed independent, the average value of  $E_A E_B$  is zero. By definition

$$(\Delta Z)^2 = \langle E_Z^2 \rangle, \quad (\Delta A)^2 = \langle E_A^2 \rangle, \quad (\Delta B)^2 = \langle E_B^2 \rangle. \quad (4.15)$$

Therefore

$$(\Delta Z)^2 = c_A^2 (\Delta A)^2 + c_B^2 (\Delta B)^2. \quad (4.16)$$

We can now state the general rule. Let  $Z$  be a known function of  $A, B, C, \dots$ . Let the standard error in  $A$  be  $\Delta A$  and so on. Then the standard error  $\Delta Z$  in  $Z$  is given by

$$(\Delta Z)^2 = (\Delta Z_A)^2 + (\Delta Z_B)^2 + (\Delta Z_C)^2 + \dots, \quad (4.17)$$

where

$$\Delta Z_A = \left( \frac{\partial Z}{\partial A} \right) \Delta A \quad \text{and so on.} \quad (4.18)$$

The expressions for  $\Delta Z$  for some common relations between  $Z$  and  $A, B$  are given in Table 4.1.

## 4.2 The straight line – method of least squares

**(a) Introduction.** In an experiment it is often the case that one quantity  $y$  is a function of another quantity  $x$ , and measurements are made of values of  $x$  and  $y$ . The values are then plotted on a graph, and we try to find a curve corresponding to some algebraic function  $y = y(x)$  which passes as closely as possible through the points. We shall consider only the case where the function is the straight line

$$y = mx + c. \quad (4.19)$$

The problem is to calculate the values of the parameters  $m$  and  $c$  for the best straight line through the points.

The straight-line relation covers a great range of physical situations. In fact we usually try to plot the graph so that the expected relationship is a straight line. For example, if we expect the refractive index  $\mu$  of a certain glass to be related to the wavelength  $\lambda$  of the light by the equation

$$\mu = a + b/\lambda^2, \quad (4.20)$$

we plot  $\mu$  against  $1/\lambda^2$ .

The standard statistical method for calculating the best values of  $m$  and  $c$  is the method of least squares.

**(b) Theory of the method.** Suppose there are  $n$  pairs of measurements  $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$  – Fig. 4.2. Assume that the errors are entirely in the  $y$  values.\* For a given pair of values for  $m$  and  $c$ , the deviation of the  $i$ th reading is

$$y_i - mx_i - c. \quad (4.21)$$

The best values of  $m$  and  $c$  are taken to be those for which

$$S = \sum (y_i - mx_i - c)^2 \quad (4.22)$$

is a minimum<sup>†</sup> – hence the name *method of least squares*.

\* The analysis for the case when there are errors in both the  $x$  and  $y$  variables is much more complicated – see Guest 1961, p. 128 – but the resulting straight line is usually quite close to that given by the present calculation – see exercise 4.4.

† The points are assumed to have equal weights. The case of unequal weights is discussed in section 4.4.

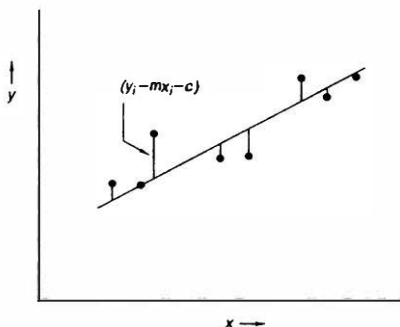


Fig. 4.2. Method of least squares. The best line through the points is taken to be the one for which  $\sum (y_i - mx_i - c)^2$  is a minimum.

The principle of minimizing the sum of the squares of the deviations was first suggested by Legendre in 1806. We have already seen that in the case of a single observable the principle gives the mean as the best value.

From (4.22)

$$\frac{\partial S}{\partial m} = -2 \sum x_i (y_i - mx_i - c) = 0, \quad (4.23)$$

$$\frac{\partial S}{\partial c} = -2 \sum (y_i - mx_i - c) = 0. \quad (4.24)$$

Therefore the required values of  $m$  and  $c$  are obtained from the two simultaneous equations

$$m \sum x_i^2 + c \sum x_i = \sum x_i y_i, \quad (4.25)$$

$$m \sum x_i + cn = \sum y_i. \quad (4.26)$$

The last equation shows that the best line goes through the point

$$\bar{x} = \frac{1}{n} \sum x_i, \quad \bar{y} = \frac{1}{n} \sum y_i, \quad (4.27)$$

i.e. through the centre of gravity of all the points. From (4.25) and (4.26)

$$m = \frac{\sum (x_i - \bar{x}) y_i}{\sum (x_i - \bar{x})^2}, \quad (4.28)$$

$$c = \bar{y} - m\bar{x}. \quad (4.29)$$

When the best values of  $m$  and  $c$  are inserted into (4.21), the deviations become the residuals

$$d_i = y_i - mx_i - c. \quad (4.30)$$

Estimates of the standard errors in  $m$  and  $c$  are given by

$$(\Delta m)^2 \approx \frac{1}{D} \frac{\sum d_i^2}{n-2}, \quad (4.31)$$

$$(\Delta c)^2 \approx \left( \frac{1}{n} + \frac{\bar{x}^2}{D} \right) \frac{\sum d_i^2}{n-2}, \quad (4.32)$$

$$D = \sum (x_i - \bar{x})^2. \quad (4.33)$$

These results are proved in Appendix C.

If we require the best line that passes through the origin, the value of  $m$  is given by (4.25) with  $c = 0$ :

$$m = \frac{\sum x_i y_i}{\sum x_i^2}. \quad (4.34)$$

An estimate of its standard error is given by

$$(\Delta m)^2 \approx \frac{1}{\sum x_i^2} \frac{\sum d_i^2}{n-1}. \quad (4.35)$$

(c) **Programmable calculator or computer.** A programmable calculator or computer may be used to calculate  $m$ ,  $c$ ,  $\Delta m$ ,  $\Delta c$ . The expressions for these quantities in equations (4.28)–(4.33) are not suitable for this purpose as they stand, because the values of  $\bar{x}$ ,  $\bar{y}$ ,  $m$ , and  $c$  (and therefore of  $d_i$ ) are not known until all the numbers are fed in. However, just as in the calculation of  $\sigma$  (section 3.4 c), the expressions may be recast into a form that avoids this difficulty. Put

$$E = \sum (x_i - \bar{x})(y_i - \bar{y}) = \sum (x_i - \bar{x})y_i, \quad (4.36)$$

$$F = \sum (y_i - \bar{y})^2. \quad (4.37)$$

From (4.28)

$$m = \frac{E}{D}. \quad (4.38)$$

Now

$$\begin{aligned} \sum d_i^2 &= \sum (y_i - mx_i - c)^2 \\ &= \sum [(y_i - \bar{y}) - m(x_i - \bar{x})]^2 \\ &= \sum (y_i - \bar{y})^2 - 2m \sum (x_i - \bar{x})(y_i - \bar{y}) + m^2 \sum (x_i - \bar{x})^2 \\ &= F - 2mE + m^2 D \\ &= F - \frac{E^2}{D}. \end{aligned} \quad (4.39)$$

From (4.31), (4.32), and (4.39)

$$(\Delta m)^2 \approx \frac{1}{n-2} \frac{DF - E^2}{D^2}, \quad (4.40)$$

$$(\Delta c)^2 \approx \frac{1}{n-2} \left( \frac{D}{n} + \bar{x}^2 \right) \frac{DF - E^2}{D^2}. \quad (4.41)$$

Equations (4.33), (4.36), and (4.37) may be written in the form

$$D = \sum x_i^2 - \frac{1}{n} (\sum x_i)^2, \quad (4.42)$$

$$E = \sum x_i y_i - \frac{1}{n} \sum x_i \sum y_i, \quad (4.43)$$

$$F = \sum y_i^2 - \frac{1}{n} (\sum y_i)^2. \quad (4.44)$$

The expressions for  $m$ ,  $c$ ,  $\Delta m$ , and  $\Delta c$  show that these quantities depend only on  $n$ , and the five sums

$$\sum x_i, \sum y_i, \sum x_i^2, \sum x_i y_i, \sum y_i^2. \quad (4.45)$$

The calculator or computer calculates the sums from the  $n$  pairs of  $x_i, y_i$  values, and then evaluates  $D, E$ , and  $F$  from (4.42)–(4.44). The quantities  $m, c, \Delta m$ , and  $\Delta c$  are then obtained from (4.38), (4.29), (4.40), and (4.41).

**(d) Worked example.** We consider an experiment to test a relation between the resistance of a semiconductor and its temperature. The semiconductor is a sample of fairly pure silicon, and a simple theory suggests that the resistance  $R$  depends on the thermodynamic temperature  $T$  according to the relation

$$R = R_0 \exp(T_0/T), \quad (4.46)$$

where  $R_0$  and  $T_0$  are constants. The resistance is measured by connecting the sample in series with a standard resistor of fixed resistance  $R_s$ , and measuring the voltage  $V_1$  across the sample and  $V_2$  across the standard when the same current passes through the two elements. Then

$$R = R_s \frac{V_1}{V_2}. \quad (4.47)$$

The temperature of the sample is measured by a nickel–chromium/copper–nickel thermocouple.\*

From (4.46)

$$\ln R = \ln R_0 + \frac{T_0}{T}. \quad (4.48)$$

So if the relation (4.46) holds, a plot of  $\ln R$  against  $1/T$  will be a straight line with slope  $T_0$ .

The calculation is readily done on a computer with a spreadsheet such as Excel<sup>®</sup>. In the description that follows it is assumed that you are familiar with the basic operation of a spreadsheet, i.e. how to create the necessary formulae, copy them from one cell to another, and so on.

\* The experiment is currently part of the first-year practical course in the Cavendish Laboratory. The theoretical relations in (4.46) and (4.52) are discussed in Kittel 1996, p. 220.

Table 4.2. Variation of the resistance of a silicon sample with temperature.  $x = 10^3 \text{ K}/T$ ,  $y = \ln(R/\Omega)$

$T/\text{K}$	$R/\Omega$	$x$	$y$
570.6	148.1	1.752	4.998
555.9	202.6	1.799	5.311
549.4	227.1	1.820	5.425
544.1	255.1	1.838	5.542
527.3	362.0	1.897	5.892
522.2	406.1	1.915	6.007
513.1	502.5	1.949	6.220
497.6	750.1	2.010	6.620
484.9	1026.7	2.062	6.934

Table 4.3. Example of spreadsheet layout for calculation of best straight line  $y = mx + c$ . The values of  $x$  and  $y$  are copied from Table 4.2

	$x$	$y$	$x^2$	$xy$	$y^2$	$d$
	1.752	4.998	3.07	8.76	24.98	-0.006
	1.799	5.311	3.24	9.55	28.21	0.016
	1.820	5.425	3.31	9.88	29.44	-0.001
	1.838	5.542	3.38	10.18	30.71	0.006
	1.897	5.892	3.60	11.17	34.71	-0.010
	1.915	6.007	3.67	11.50	36.08	-0.011
	1.949	6.220	3.80	12.12	38.68	-0.009
	2.010	6.620	4.04	13.30	43.83	0.014
	2.062	6.934	4.25	14.30	48.08	0.000
Sum	17.042	52.949	32.35	100.78	314.72	-1.5E-14
$n$	9		$m$		6.225	
$D$	0.0829		$\Delta m$		0.038	
$E$	0.5159		$c$		-5.905	
$F$	3.2124		$\Delta c$		0.073	

It is convenient to do the calculation in two stages. First we write a program, i.e. a set of formulae in the spreadsheet, to calculate  $x$  and  $y$  from the experimental data. The output of this stage is shown in Table 4.2. The measured values of  $T$  and  $R$  are entered in the first two columns of the table. Put

$$x = 10^3 \text{ K}/T, \quad y = \ln(R/\Omega). \quad (4.49)$$

Note that the dimensions of  $T$  are [K], so  $x$  is a dimensionless quantity. The factor  $10^3$  is inserted in the definition of  $x$  to give values of the order of unity – see section 10.6.

The second stage is to copy the values of  $x$  and  $y$  into another program that does the actual least-squares calculation. The output of this stage is shown in Table 4.3. For each pair of  $x$ ,  $y$  values the program calculates the values of  $x^2$ ,  $xy$ ,

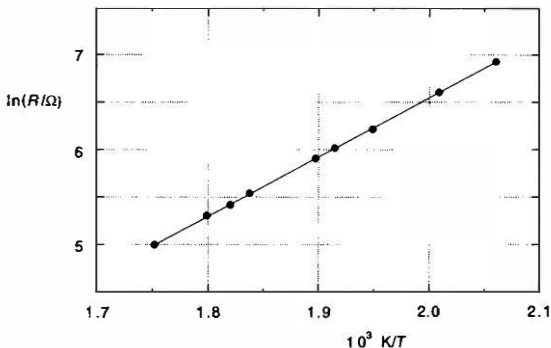


Fig. 4.3. Plot of  $y = \ln(R/\Omega)$  versus  $x = 10^3 K/T$  for the values in Table 4.2.

and  $y^2$ , counts the number of  $x, y$  pairs in the data, and evaluates the five sums in (4.45), which are shown at the foot of each column. The values of  $D$ ,  $E$ , and  $F$ , and finally those of  $m$ ,  $c$ ,  $\Delta m$ , and  $\Delta c$  are then obtained from the equations in the previous section.

It is good practice to evaluate the deviations

$$d_i = y_i - (mx_i + c). \quad (4.50)$$

These are listed in the last column of Table 4.3, and their sum given at the bottom. Theoretically the sum is zero – see exercise 4.6. It will not be exactly zero in the spreadsheet owing to rounding errors in computer arithmetic, but if it is not very small there is a mistake in the calculation. The sum in Table 4.3 is  $-1.5\text{E-}14$ , i.e.  $-1.5 \times 10^{-14}$ .

The values of  $x$  and  $y$  are plotted in Fig. 4.3, together with the best straight line. The values are evidently a good fit to a straight line, showing that, within experimental error, the relation (4.46) holds over the range of temperature of the experiment. Due to the factor  $10^3$  in the definition of  $x$ , the value of  $T_0$  is  $10^3$  times that of  $m$ . Thus, taking the error to one significant digit, we have the result

$$T_0 = (6230 \pm 40) \text{ K}. \quad (4.51)$$

The intercept  $c$  is equal to  $\ln R_0$ , but this depends on the particular silicon sample and is not of interest in the experiment.

The theory shows that the energy gap  $E_g$  between the valence band and the conduction band of a semiconductor is related to  $T_0$  by

$$E_g = 2kT_0, \quad (4.52)$$

where  $k$  is the Boltzmann constant. The value of  $T_0$  in (4.51), together with the values of  $k$  and  $e$ , the charge on the electron, given on p. 192, give

$$E_g = (1.073 \pm 0.007) \text{ eV}, \quad (4.53)$$

which may be compared with a value of about 1.06 eV for pure silicon.

It may be noted that by doing the calculation in two stages we have in the second program a general least-squares fit for a straight line. Having saved this program, we only need feed in a set of  $x, y$  values, and we have the values of  $m, c, \Delta m$ , and  $\Delta c$ .

### 4.3 The straight line - points in pairs

In the absence of a programmable calculator or a computer, the least-squares calculation is laborious. The following method provides a simple alternative that is often adequate for the purpose. It is particularly useful when the  $x$  values are equally spaced.

In order to illustrate the method, suppose that we have 8 points that lie approximately on a straight line, and we require the best value of the slope  $m$  and the error in it. Let the points be numbered in order from 1 to 8 - see Fig. 4.4. Consider points 1 and 5; they determine a straight line and hence a value for the slope. Pairing the points in this way we obtain four values for the slope. We take their mean  $\bar{m}$  as the best value of  $m$  and find its standard error in the usual way.

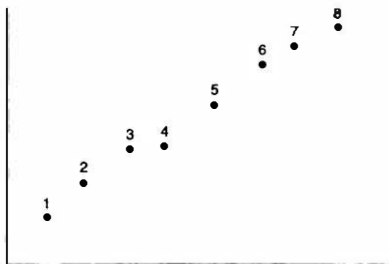


Fig. 4.4. Simple method of estimating slope of best line. Each pair of points 1-5, 2-6, etc. gives a value of the slope. The mean is taken as the best value.

The method will give a reasonable result only if the quantities  $(x_5 - x_1)$ ,  $(x_6 - x_2)$ ,  $(x_7 - x_3)$ ,  $(x_8 - x_4)$  are roughly equal. Otherwise, the four values of the slope do not have equal weight.

The best line given by this method is the one with slope  $\bar{m}$  that passes through the point  $\bar{x}, \bar{y}$ . (We have already seen that the line given by the method of least squares passes through this point.) However, the method is mainly used when only the slope is required.



### 4.4 Weighting of results

Suppose we measure a quantity a certain number of times, say 10, and obtain the values  $x_1, x_2, \dots, x_{10}$ . Suppose further that we divide the measurements into two sets and calculate the mean of each. For example, we might make 7 of the measurements in the morning and calculate their mean

$$z_1 = \frac{1}{7}(x_1 + x_2 + \dots + x_7). \quad (4.54)$$

Then we might make the other 3 in the afternoon and calculate their mean

$$z_2 = \frac{1}{3}(x_8 + x_9 + x_{10}). \quad (4.55)$$

The best value from all 10 measurements is

$$\bar{z} = \frac{1}{10}(x_1 + x_2 + \dots + x_{10}), \quad (4.56)$$

and obviously it is not given by taking the simple mean of  $z_1$  and  $z_2$ . If we wish to calculate it from these two quantities, it is given by

$$\bar{z} = \frac{7z_1 + 3z_2}{10}. \quad (4.57)$$

The numbers 7 and 3 are termed the *weights* or *relative weights* of the quantities  $z_1$  and  $z_2$ .

In general, if we have a set of values  $z_1, z_2, \dots, z_N$  with relative weights  $w_1, w_2, \dots, w_N$ , then the best value of the quantity is

$$\bar{z} = \frac{\sum w_i z_i}{\sum w_i}. \quad (4.58)$$

If all the  $w$ s are multiplied by a constant, the value of  $\bar{z}$  is unchanged, so it is only the ratios of the  $w$ s that matter.

Suppose now that we have  $N$  measurements of the quantity  $z$ , each measurement having its own standard error, i.e. we have

$$z_1 \pm \Delta z_1, z_2 \pm \Delta z_2, \dots, z_N \pm \Delta z_N.$$

What weight should we give to each  $z_i$  in order to obtain the best value of  $z$  from all the measurements? The answer is provided by the simple example at the beginning of the section. We saw that if  $z_i$  is the mean of  $n_i$  original values, then its weight  $w_i$  is proportional to  $n_i$ . This assumes that all the original values have the same weight, that is, that they all come from the same distribution characterized by a certain standard error  $\sigma$ . We therefore *imagine* each  $z_i$  in the above set is the mean of  $n_i$  original values taken from a distribution of standard error  $\sigma$ , and give it weight  $n_i$ .

We do not know the value of  $\sigma$ ; in fact we can choose it quite arbitrarily but, having fixed on a value, we use the result

$$\Delta z_i = \frac{\sigma}{\sqrt{n_i}} \quad (4.59)$$

to obtain  $n_i$ . So

$$w_i = n_i = \frac{\sigma^2}{(\Delta z_i)^2} \quad (4.60)$$

The standard error in  $\bar{z}$  is  $\sigma/(\sum n_i)^{1/2}$ .

From (4.58) and (4.60) the best value of  $\bar{z}$  and its standard error are

$$\frac{\sum (1/\Delta z_i)^2 z_i}{\sum (1/\Delta z_i)^2} \pm \frac{1}{\left[ \sum (1/\Delta z_i)^2 \right]^{1/2}} \quad (4.61)$$

Both these expressions are independent of the value of  $\sigma$  as of course they must be.

In section 4.2 we gave the method of least squares for finding the best straight line through a set of points of equal weight. The generalization to the case of unequal weights is readily made. If  $w_i$  is the weight of the pair of values  $x_i, y_i$ , then it is necessary to minimize the quantity

$$S_w = \sum w_i (y_i - mx_i - c)^2 \quad (4.62)$$

The equations for  $m$  and  $c$  become

$$m \sum w_i x_i^2 + c \sum w_i x_i = \sum w_i x_i y_i \quad (4.63)$$

$$m \sum w_i x_i + c \sum w_i = \sum w_i y_i \quad (4.64)$$

The expressions for  $m$  and  $c$  and their standard errors are given in the summary on p. 40.

It can be seen that the expressions depend only on the six sums

$$\sum w_i, \quad \sum w_i x_i, \quad \sum w_i y_i, \quad \sum w_i x_i^2, \quad \sum w_i x_i y_i, \quad \sum w_i y_i^2. \quad (4.65)$$

A spreadsheet calculation of  $m$ ,  $c$ ,  $\Delta m$ , and  $\Delta c$  for unequal weights is thus a straightforward extension of that for equal weights. In the latter case we had in Table 4.3, p. 34, columns headed

$$x \quad y \quad x^2 \quad xy \quad y^2 \quad d \quad (4.66)$$

For unequal weights the column headings are

$$x \quad y \quad w \quad wx \quad wy \quad wx^2 \quad wxy \quad wy^2 \quad wd \quad (4.67)$$

The experimental values of  $x_i, y_i, w_i$  are entered in the first three columns. The spreadsheet is programmed to calculate the values in the next five columns and to evaluate their sums, together with the sum of the weights. After the values of  $m$  and  $c$  have been obtained, the weighted deviations

$$w_i d_i = w_i (y_i - mx_i - c) \quad (4.68)$$

are evaluated and entered in the last column. As in the case of equal weights you should check that  $\sum w_i d_i$  is very small.

## Summary of equations for the best straight line by the method of least squares

$n$  points  $x_i, y_i$

**Equal weights**

General line  $y = mx + c$

$$m = \frac{E}{D} \quad c = \bar{y} - m\bar{x}$$

$$(\Delta m)^2 \approx \frac{1}{n-2} \frac{\sum d_i^2}{D} = \frac{1}{n-2} \frac{DF - E^2}{D^2}$$

$$(\Delta c)^2 \approx \frac{1}{n-2} \left( \frac{D}{n} + \bar{x}^2 \right) \frac{\sum d_i^2}{D} = \frac{1}{n-2} \left( \frac{D}{n} + \bar{x}^2 \right) \frac{DF - E^2}{D^2}$$

$$D = \sum x_i^2 - \frac{1}{n} (\sum x_i)^2$$

$$E = \sum x_i y_i - \frac{1}{n} \sum x_i \sum y_i$$

$$F = \sum y_i^2 - \frac{1}{n} (\sum y_i)^2$$

$$\bar{x} = \frac{1}{n} \sum x_i \quad \bar{y} = \frac{1}{n} \sum y_i$$

$$d_i = y_i - mx_i - c$$

Line through origin  $y = mx$

$$m = \frac{\sum x_i y_i}{\sum x_i^2}$$

$$(\Delta m)^2 \approx \frac{1}{n-1} \frac{\sum d_i^2}{\sum x_i^2} = \frac{1}{n-1} \frac{\sum x_i^2 \sum y_i^2 - (\sum x_i y_i)^2}{(\sum x_i^2)^2}$$

$$d_i = y_i - mx_i$$

**Unequal weights***General line*  $y = mx + c$ 

$$m = \frac{E}{D} \quad c = \bar{y} - m\bar{x}$$

$$(\Delta m)^2 \approx \frac{1}{n-2} \frac{\sum w_i d_i^2}{D} = \frac{1}{n-2} \frac{DF - E^2}{D^2}$$

$$(\Delta c)^2 \approx \frac{1}{n-2} \left( \frac{D}{\sum w_i} + \bar{x}^2 \right) \frac{\sum d_i^2}{D} = \frac{1}{n-2} \left( \frac{D}{\sum w_i} + \bar{x}^2 \right) \frac{DF - E^2}{D^2}$$

$$D = \sum w_i x_i^2 - \frac{1}{\sum w_i} (\sum w_i x_i)^2$$

$$E = \sum w_i x_i y_i - \frac{1}{\sum w_i} \sum w_i x_i \sum w_i y_i$$

$$F = \sum w_i y_i^2 - \frac{1}{\sum w_i} (\sum w_i y_i)^2$$

$$\bar{x} = \frac{\sum w_i x_i}{\sum w_i} \quad \bar{y} = \frac{\sum w_i y_i}{\sum w_i}$$

$$d_i = y_i - mx_i - c$$

*Line through origin*  $y = mx$ 

$$m = \frac{\sum w_i x_i y_i}{\sum w_i x_i^2}$$

$$(\Delta m)^2 \approx \frac{1}{n-1} \frac{\sum w_i d_i^2}{\sum w_i x_i^2} = \frac{1}{n-1} \frac{\sum w_i x_i^2 \sum w_i y_i^2 - (\sum w_i x_i y_i)^2}{(\sum w_i x_i^2)^2}$$

$$d_i = y_i - mx_i$$

## Exercises

- 4.1 In the following examples,  $Z$  is a given function of the independently measured quantities  $A, B, \dots$ . Calculate the value of  $Z$  and its standard error  $\Delta Z$  from the given values of  $A \pm \Delta A, B \pm \Delta B, \dots$

(a)  $Z = A^2, \quad A = 25 \pm 1.$

(b)  $Z = A - 2B, \quad A = 100 \pm 3, \\ B = 45 \pm 2.$

(c)  $Z = \frac{A}{B}(C^2 + D^2), \quad A = 0.100 \pm 0.003, \\ B = 1.00 \pm 0.05, \\ C = 50.0 \pm 0.5, \\ D = 100 \pm 8.$

(d)  $Z = A \ln B, \quad A = 10.00 \pm 0.06, \\ B = 100 \pm 2.$

(e)  $Z = 1 - \frac{1}{A}, \quad A = 50 \pm 2.$

- 4.2 The volume  $V$  of a rectangular block is determined by measuring the lengths  $l_x, l_y, l_z$  of its sides. From the scatter of the measurements a standard error of 0.01% is assigned to each dimension. What is the standard error in  $V$  (a) if the scatter is due to errors in setting and reading the measuring instrument, and (b) if it is due to temperature fluctuations?

- 4.3 A weight  $W$  is suspended from the centre of a steel bar which is supported at its ends, and the deflection at the centre is measured by means of a dial height indicator whose readings are denoted by  $y$ . The following values are obtained:

$W/\text{kg}$	$y/\mu\text{m}$
0	1642
$\frac{1}{2}$	1483
1	1300
$1\frac{1}{2}$	1140
2	948
$2\frac{1}{2}$	781
3	590
$3\frac{1}{2}$	426
4	263
$4\frac{1}{2}$	77

- (a) Plot the points on a graph and draw the best line by eye. Make an intelligent guess of the standard error in the slope by placing a transparent rule along the points and seeing what might be reasonable limits for the line.
- (b) Calculate the best value of the slope and its standard error by the method of least squares, and compare the results with your estimates in (a).
- (c) Calculate the best value of the slope and its standard error by the method of

points in pairs, and draw the line with this slope through the point  $\bar{x}$ ,  $\bar{y}$ . Compare these results with those of (b).

- 4.4 The zener diode is a semiconductor device with the property that its resistance drops suddenly to almost zero when the reverse bias voltage exceeds a critical value  $V_z$ , which depends on the temperature  $T$  of the diode. The value of  $V_z$  is of the order of volts, but the temperature coefficient  $dV_z/dT$  is only a few millivolts per  $^{\circ}\text{C}$  in the temperature range  $20\text{--}80^{\circ}\text{C}$ . Therefore, to measure  $dV_z/dT$  precisely, a constant reference voltage is subtracted from  $V_z$ , and the resulting voltage  $V$  is measured directly on a digital multimeter. The following values are obtained for a particular zener diode:

$T/^{\circ}\text{C}$	$V/\text{mV}$	$T/^{\circ}\text{C}$	$V/\text{mV}$
24.0	72.5	50.0	139
30.0	93	56.2	156.5
37.6	107	61.0	171
40.0	116	64.6	178
44.1	127	73.0	198.5

Treat the data in the same way as in parts (a), (b), and (c) of exercise 4.3, assuming in part (b) that the temperature measurements are free from error. This assumption is probably not correct, so repeat the least-squares calculation, assuming that the voltage measurements are free from error, and compare the two values of  $dV_z/dT$ .

- 4.5 The free neutron is an unstable particle and decays into a proton, an electron, and an antineutrino. If there are  $N_0$  neutrons at time  $t = 0$ , the number of undecayed neutrons at time  $t$  is  $N_0 \exp(-t/\tau)$ , where  $\tau$  is a constant known as the *neutron lifetime*. The results of the four most precise measurements of  $\tau$  given by Yerozolimsky 1994 are

Year	$\tau/\text{s}$	$\Delta\tau/\text{s}$
1989	887.6	3.0
1990	893.5	5.3
1992	888.4	3.3
1993	882.6	2.7

Calculate the weighted mean of the values of  $\tau$  and its standard error.

- 4.6 Show that, for the best straight line  $y = mx + c$  calculated by the method of least squares, the deviations  $d_i$  in (4.30) satisfy the relations

$$\sum d_i = 0,$$

$$\sum x_i d_i = 0.$$

- 4.7 In section 4.2 we obtained the value of  $m$  for the best line through the origin by the method of least squares. Another method is to take the weighted mean of the values  $m_i = y_i/x_i$ . Show that this gives the same result as (4.34).

### 5.1 Error calculations in practice

We are now in a position to estimate the standard errors for a large class of experiments. Let us briefly recapitulate. The final quantity  $Z$  is a function of the primary quantities  $A, B, C, \dots$  which are either measured directly or are the slopes or intercepts of straight lines drawn through points representing directly measured quantities.

If the quantity is measured directly, we take the mean of several values to be the best value and obtain its standard error by the method given in chapter 3. (During the present chapter we shall drop the word 'standard' in 'standard error'. We shall not be considering the *actual* error in a measured quantity, and the word 'error' will refer to the standard error, i.e. the standard deviation of the distribution of which the quantity is a member.) If the quantity is the slope or intercept in a straight line, its value and error are obtained either from the method of least squares or from the method of taking the points in pairs.

The best value of  $Z$  is calculated from the best values of the primary quantities, and its error is obtained from their errors by the rules given in Table 4.1, or in general from (4.17) and (4.18).

There are often a large number of primary quantities to be measured, and it might be thought that the calculation of the error in each one and the subsequent calculation of the error in  $Z$  would be a laborious process. And with many students it is indeed. They calculate the standard deviation automatically for every set of measurements, and combine all the errors irrespective of their magnitudes according to the formal rules, involving themselves in elaborate calculations and ending up with an error calculated to a meaningless number of decimal places, which is usually wrong by several orders of magnitude due to various arithmetical slips on the way.

To see what is required in practice, let us first remember *why* we estimate errors. It is to provide a measure of the significance of the final result. The use made of the error is seldom based on such precise calculation that we need its value to better than 1 part in 4. Often we are interested in the error to much less precision, perhaps only to within a factor of 2. However, let us take 1 part in 4 as an arbitrary but adequate degree of precision for the final error.

(a) *Combining errors.* If we look at the equation for combining errors (4.17), we see that, owing to the procedure of squaring the terms, one error is often negligible compared with another. Consider the case

$$Z = A + B, \quad (5.1)$$

and let  $\Delta A = 2$  and  $\Delta B = 1$ . From Table 4.1, (i)

$$\Delta Z = (2^2 + 1^2)^{\frac{1}{2}} = 2.24. \quad (5.2)$$

So even though  $\Delta B$  is as much as one-half of  $\Delta A$ , ignoring  $\Delta B$  altogether and putting  $\Delta Z \approx \Delta A = 2$  makes a difference of only about 1 part in 8 in the final error. If  $Z$  is the sum of several quantities, ignoring errors that are one-half of the largest error may be rather drastic, but we shall nearly always be justified in ignoring an error less than one-third of the largest error.

We may notice also the situation when the quantities themselves differ greatly in magnitude. For example, suppose in (5.1) that  $B$  is some small correction term and we have values

$$\begin{aligned} A &= 100 \pm 6 \\ B &= 5 \pm ? \end{aligned}$$

The error in  $B$  will be negligible unless it is as much as 3, but such an error amounts to 60% of  $B$ ; so the quantity will have to be measured very roughly indeed if its error is to contribute.

In the case of multiplication and division – Table 4.1, (ii) – we add the squares, not of the errors themselves, but of the fractional errors. So in this case, all fractional errors less than about one-third of the largest fractional error may be neglected.

(b) *Contributing and non-contributing errors.* With these considerations in mind let us go back to the estimation of the errors in the primary quantities. We may call a quantity *contributing* or *non-contributing* according to whether or not its error contributes appreciably to the final error. A quantity may be non-contributing either because it is measured relatively precisely or because it is added to a much larger quantity.

If we suspect that a quantity is non-contributing, it is sufficient to estimate its error very roughly, provided the estimate is on the high side. The reason for this condition is obvious. It ensures that we do not omit the error unjustifiably. If the inflated error is negligible we are quite safe. If not, we must go back to the measurements and work out the error more carefully.

For example, suppose the results of successive weighings of an object are:

$$\begin{aligned} &50.3853 \text{ g} \\ &50.3846 \\ &50.3847 \\ &50.3849. \end{aligned}$$



We take the best value of the weight to be

$$50.3849 \pm 0.0003 \text{ g.}$$

We expect this set of measurements to be much more precise than several others in the particular experiment and we therefore estimate an error simply by inspecting the measurements. The value 0.0003 encompasses 3 out of the 4 individual readings, so it is almost certain to be an overestimate of the error in the mean.

(c) *Discrete readings.* Another case where a common-sense estimate of the error should be made is when the readings are in digital form or are taken to the nearest scale division of an instrument, and show little or no spread. Consider the following set of measurements made with a metre rule:

$$325, 325, 325, 325\frac{1}{2}, 325, 325 \text{ mm.}$$

The most one can say is that the measured quantity is  $325 \pm \frac{1}{2}$  mm or  $325 \pm \frac{1}{4}$  mm.\* If a better value of the quantity and its error are required, they will not be obtained by more arithmetic, nor by more measurements of the same kind. Either the scale should be estimated to  $\frac{1}{10}$  mm as the measurements are made, or a more precise instrument such as a cathetometer should be used.

(d) *Systematic errors.* So far we have confined the discussion to the estimation of random errors. And this is all that is needed in the majority of experiments. Any systematic error that we know about should be corrected and hence eliminated – or at least rendered negligible. Normally we would reduce it to a level small compared with the random errors. So it would be non-contributing and would not enter the error calculation.

The occasional situation when residual systematic errors are not small compared with random errors should be discussed and treated on its merits. One way of proceeding is to try to estimate, for each systematic error, something equivalent to a standard error, that is to say, a quantity such that we think there are about 2 chances in 3 that the true value lies within the quoted range. For example, we might estimate – or make an intelligent guess of – an upper limit, and then divide it by 2. (This may seem rough and ready, but a crude estimate is better than none at all.) All the errors are then combined as though they were random and independent. When this is done, it is good practice to make quite clear how much of the final error is due to the actual random error and how much to the various systematic errors.

(e) *The final quoted error.* We may sum up as follows. Systematic errors are eliminated as far as possible. The random errors in contributing quantities are calculated by an appropriate statistical method. Other errors are estimated

\* It is not unknown for students to solemnly feed these numbers into their calculators, arriving at the result  $325.08 \pm 0.08$  mm.

roughly, the estimates being slightly on the high side. A check – which can often be done mentally – is made that these errors are in fact negligible. The contributing errors are then combined according to the rules of Table 4.1 to give the final quoted error. This quantity represents our best estimate of the standard deviation of the distribution of results that would be obtained if the entire experiment were repeated many times with the same or similar apparatus. It is thus a measure of the overall *reproducibility* of the result.

Some experimenters, having obtained the overall error in the usual way, then proceed to enlarge it by an arbitrary factor to take account of possible, but unknown, sources of systematic error. This is highly undesirable. It is difficult for anyone to make use of these subjective overestimates. You should estimate the error as honestly as you are able and leave it at that. If it subsequently turns out that the 'true' value of the quantity being measured is several times your estimated error away from the value you have obtained, you may or may not be held at fault. But you must not arbitrarily double or treble the error as a kind of safety measure to prevent the situation arising. Quite apart from the confusion caused by the uncertain significance of the final error, the procedure may obscure genuine discrepancies between one experimental result and another, or between theory and experiment.

It is conventional to quote the final error in absolute terms, and not as a fraction of the final value of the quantity being measured, although it is often useful to give this fraction or percentage in addition. The final value of the quantity and its error should be given to the same number of digits, which should not be more than are meaningful. In general this corresponds to an error of one significant digit, though, if this digit is 1 or 2 a second digit might be given. The fact that we do not want an estimate of the final error more precise than this means that the whole error calculation should be done only to one or at the most two significant digits.

## 5.2 Complicated functions

The evaluation of quantities of the type  $\partial Z/\partial A$  in (4.18) is sometimes quite laborious. As an example consider the measurement of the refractive index  $\mu$  of a glass prism by measuring the angle  $A$  of the prism and the angle  $D$  of minimum deviation. The refractive index is obtained from the equation

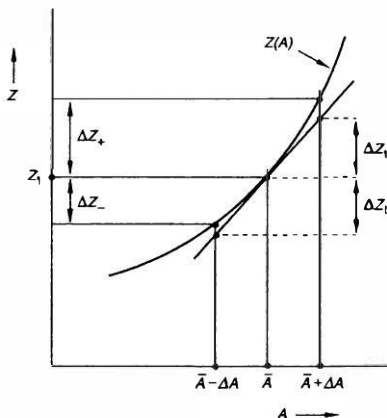
$$\mu = \frac{\sin \frac{1}{2}(A + D)}{\sin \frac{1}{2}A}. \quad (5.3)$$

The error in  $\mu$  is given by

$$(\Delta\mu)^2 = (\Delta\mu_A)^2 + (\Delta\mu_D)^2. \quad (5.4)$$

$\Delta\mu_A$  is the error in  $\mu$  due to the error  $\Delta A$  in  $A$  and is given by

$$\Delta\mu_A = \left( \frac{\partial\mu}{\partial A} \right) \Delta A. \quad (5.5)$$

Fig. 5.1. Relation between different estimates of  $\Delta Z$ .

Similarly for  $\Delta\mu_D$ .

The expressions for  $\partial\mu/\partial A$  and  $\partial\mu/\partial D$  are

$$\frac{\partial\mu}{\partial A} = \frac{1}{2} \frac{\cos \frac{1}{2}(A+D)}{\sin \frac{1}{2}A} - \frac{1}{2} \frac{\sin \frac{1}{2}(A+D)}{\sin \frac{1}{2}A \tan \frac{1}{2}A}, \quad (5.6)$$

$$\frac{\partial\mu}{\partial D} = \frac{1}{2} \frac{\cos \frac{1}{2}(A+D)}{\sin \frac{1}{2}A}, \quad (5.7)$$

These expressions have to be evaluated at  $A = \bar{A}$ , the measured value of  $A$ , and  $D = \bar{D}$ , the measured value of  $D$ . And, provided we do the arithmetic correctly and remember to express  $\Delta A$  and  $\Delta D$  in radians, we shall get the right answer for  $\Delta\mu_A$  and  $\Delta\mu_D$ .

However, there is a quicker method. Consider the significance of  $\Delta\mu_A$ . It is the change in the value of  $\mu$  when  $A$  changes by an amount  $\Delta A$ , the value of  $D$  remaining constant. So it may be obtained by calculating  $\mu$  from (5.3), first for  $A = \bar{A}$ ,  $D = \bar{D}$  and then for  $A = \bar{A} + \Delta A$ ,  $D = \bar{D}$ . The difference is  $\Delta\mu_A$ . Similarly  $\Delta\mu_D$  is obtained by calculating  $\mu$  for  $A = \bar{A}$ ,  $D = \bar{D} + \Delta D$ . All we need are the sine values. We do not have to do any complicated algebra or arithmetic – fruitful sources of mistakes – nor bother to convert  $\Delta A$  and  $\Delta D$  into radians. We combine  $\Delta\mu_A$  and  $\Delta\mu_D$  in the usual way.

The fact that this method of calculating  $\Delta\mu_A$  and  $\Delta\mu_D$  is much quicker than the more formal method should not lead you to imagine that it is in any way less rigorous or exact. The two methods usually give the same answer, and when they

do not, the formal method is not valid. This may be seen from Fig. 5.1, where the results of the two methods are shown for some relation  $Z = Z(A)$ . The best value of  $A$  is  $\bar{A}$  and this corresponds to  $Z_1$ . The error  $\Delta Z_1$  obtained by the formal method corresponds to putting the tangent to the curve  $Z(A)$  at the point  $\bar{A}$ ,  $Z_1$ . The value calculated from the simpler and more direct method is  $\Delta Z_+$  in the diagram. We could equally well have calculated  $\Delta Z$  by taking the value  $\bar{A} - \Delta A$ , which would have given  $\Delta Z_-$ .

The curvature of the function  $Z(A)$  over the range  $\bar{A} \pm \Delta A$  is not usually as large as that shown in Fig. 5.1; in which case the difference between  $\Delta Z_1$ ,  $\Delta Z_+$  and  $\Delta Z_-$  is negligible. If, however, the curvature of the function is significant, then a single value of the error is misleading. Instead, both  $\Delta Z_+$  and  $\Delta Z_-$  should be calculated and the result quoted as

$$Z = Z_1 \begin{matrix} +\Delta Z_+ \\ -\Delta Z_- \end{matrix}.$$

Such refinement is seldom justified. The main point is that to calculate the error in  $Z$  due to an error in  $A$ , we cannot go wrong if we simply calculate the values of  $Z$  at the values  $\bar{A}$  and  $\bar{A} \pm \Delta A$ , with the other measured quantities constant. And often this is much quicker than the formal method.

### 5.3 Errors and experimental procedure

When the final quantity  $Z$  is related to two directly measured quantities by a function of the form

$$Z = AB \quad \text{or} \quad A/B,$$

then an error of  $x\%$  in  $A$  or  $B$  gives rise to an error of  $x\%$  in  $Z$ . So we would try to measure  $A$  and  $B$  with comparable precision, and this is true whatever the relative magnitudes of  $A$  and  $B$ . But the situation

$$Z = A + B \quad \text{or} \quad A - B$$

is quite different. Everything depends on the relative magnitudes of  $A$  and  $B$ . Look at the following example:

$$\begin{aligned} \text{Case I} \quad A &= 10000 \pm 1, \\ B &= 100 \pm 5, \\ Z = A + B &= 10100 \pm 5. \end{aligned}$$

Here  $A$  is a large, precisely known quantity.  $B$  has been measured to 5%, but the final quantity  $Z$  has been found to 0.05%. So we see that it is advantageous to start with a large, precisely known quantity and simply measure a small additional term in order to get the required quantity.

Now consider the following:

$$\begin{aligned} \text{Case II} \quad A &= 100 \pm 2, \\ B &= 96 \pm 2, \\ Z = A - B &= 4 \pm 3. \end{aligned}$$

The two directly measured quantities have been determined to 2%, but the final quantity is only known to 75%. So taking the difference between two nearly equal quantities, each of which is measured independently, is inherently disadvantageous; the final error is greatly magnified. If possible an entirely different method of measuring  $Z$  should be found.

In the next two chapters we shall give specific examples of methods devised to take advantage of the Case I situation and others devised to avoid Case II. They provide examples of the way error considerations may have a direct influence on experimental procedure.

We give one hypothetical example here. Consider the following situation. We require to measure the quantity  $Z = A/B$ . We have made a set of measurements and found

$$A = 1000 \pm 20,$$

$$B = 10 \pm 1.$$

Therefore

$$\frac{\Delta A}{A} = 2\% \quad \text{and} \quad \frac{\Delta B}{B} = 10\%,$$

whence

$$\frac{\Delta Z}{Z} = (2^2 + 10^2)^{\frac{1}{2}} = 10.2\%.$$

We have some further time available for measurements and estimate it is sufficient to reduce the error in either  $A$  or  $B$  by a factor 2. If we devote the time to  $A$ , we shall have

$$\frac{\Delta A}{A} = 1\%, \quad \text{which gives} \quad \frac{\Delta Z}{Z} = (1^2 + 10^2)^{\frac{1}{2}} = 10.0\%.$$

If we devote it to  $B$ , we shall have

$$\frac{\Delta B}{B} = 5\%, \quad \text{which gives} \quad \frac{\Delta Z}{Z} = (2^2 + 5^2)^{\frac{1}{2}} = 5.4\%.$$

So in the first case the overall error is barely changed, and in the second case it is reduced by a factor of almost 2. The moral is *always concentrate on quantities that contribute most to the final error.*

In general one should plan the experiment so that in the final result no one quantity contributes an error much greater than the others. In the present example we may suspect that the original measurements, which resulted in  $\Delta B/B$  being 5 times greater than  $\Delta A/A$ , were badly planned, and that more time should have been devoted to measuring  $B$  at the expense of  $A$ . Of course it is not always the case that additional measurements result in a reduction of the error. Nevertheless, the desirability of reducing the maximum contributing error should always be kept in mind when planning an experiment.

### Summary of treatment of errors

**Strategy**

Identify the quantity contributing most to the final error, and try to reduce it, either by making more measurements or by using a different method.

**Calculation**

An estimate of the error in the final result good to 1 part in 4 is usually adequate. So the error calculation should be done to one, or at the most two, significant digits.

Ignore all errors whose contribution to the final error is less than about a third of the largest contribution.

Combine the errors that remain.

The final error should be equal to, or larger than, the largest contribution (it cannot be less), but it will usually be only *slightly* larger. If it is much larger, you have probably made a mistake.

**Final result**

Quote the result and its error to the same number of digits.

It is often useful to give, in addition, the error as a fraction, or a percentage, of the result.

**Systematic errors**

The above refers to *random* errors. Comment on any systematic errors, and estimate an upper limit for them.

## Exercises

- 5.1 A rectangular brass bar of mass  $M$  has dimensions  $a$ ,  $b$ ,  $c$ . The moment of inertia  $I$  about an axis in the centre of the  $ab$  face and perpendicular to it is

$$I = \frac{M}{12} (a^2 + b^2).$$

The following measurements are made:

$$M = 135.0 \pm 0.1 \text{ g,}$$

$$a = 80 \pm 1 \text{ mm,}$$

$$b = 10 \pm 1 \text{ mm,}$$

$$c = 20.00 \pm 0.01 \text{ mm.}$$

What is the percentage standard error in (a) the density  $\rho$  of the material, and (b) the moment of inertia?

- 5.2 When a torsion wire of radius  $r$  and length  $l$  is fixed at one end and subjected to a couple of moment  $C$  at the other, the angular displacement  $\phi$  is given by

$$\phi = \frac{2lC}{n\pi r^4},$$

where  $n$  is the rigidity modulus of the material of the wire. The following values are obtained:

$$\phi/C = 4.00 \pm 0.12 \text{ rad N}^{-1}\text{m}^{-1},$$

$$r = 1.00 \pm 0.02 \text{ mm,}$$

$$l = 500 \pm 1 \text{ mm.}$$

Calculate the value of  $n$  and its standard error.

- 5.3 For Kater's pendulum,  $g$  is obtained from the equation

$$\frac{8\pi^2}{g} = \frac{T_1^2 + T_2^2}{H} + \frac{T_1^2 - T_2^2}{h_1 - h_2},$$

where  $T_1$  is the period of oscillation about one of the knife-edges and  $h_1$  the distance from this knife-edge to the centre of gravity of the pendulum.  $T_2$  and  $h_2$  are the corresponding quantities for the other knife-edge. The quantity  $H$  is equal to  $h_1 + h_2$  and, being the distance between the knife-edges, is measured directly.

The following measurements are made:

$$T_1 = 2.00428 \pm 0.00005 \text{ s,}$$

$$T_2 = 2.00229 \pm 0.00005 \text{ s,}$$

$$H = 1.00000 \pm 0.00004 \text{ m,}$$

$$h_1 = 0.700 \pm 0.001 \text{ m,}$$

$$h_2 = 0.300 \pm 0.001 \text{ m.}$$

Calculate the measured value of  $g$  and its standard error.

- 5.4 Calculate the value and standard error of the refractive index of the glass of a prism from the following results:

$$\text{angle of prism } A = 60^\circ 18' \pm 10'.$$

$$\text{angle of minimum deviation } D = 35^\circ 46' \pm 20'.$$

- 5.5 If a narrow collimated beam of monoenergetic  $\gamma$ -rays of intensity  $I_0$  is incident on a thin sheet of material of thickness  $x$ , the intensity of the emerging beam is given by

$$I = I_0 \exp(-\mu x),$$

where  $\mu$  is a quantity known as the linear attenuation coefficient. The following values are obtained for  $\gamma$ -rays of energy 1 MeV incident on lead:

$$I = (0.926 \pm 0.010) \times 10^{10} \text{ } \gamma\text{-rays m}^{-2} \text{ s}^{-1},$$

$$I_0 = (2.026 \pm 0.012) \times 10^{10} \text{ } \gamma\text{-rays m}^{-2} \text{ s}^{-1},$$

$$x = (10.00 \pm 0.02) \text{ mm}.$$

Calculate the value of  $\mu$  and its standard error for  $\gamma$ -rays of this energy in lead.

- 5.6 Neutrons reflected by a crystal obey Bragg's law  $n\lambda = 2d \sin \theta$ , where  $\lambda$  is the de Broglie wavelength of the neutrons,  $d$  is the spacing between the reflecting planes of atoms in the crystal,  $\theta$  is the angle between the incident (or reflected) neutrons and the atomic planes, and  $n$  is an integer. If  $n$  and  $d$  are known, the measured value of  $\theta$  for a beam of monoenergetic neutrons determines  $\lambda$ , and hence the kinetic energy  $E$  of the neutrons. If  $\theta = 11^\circ 18' \pm 9'$ , what is the fractional error in  $E$ ?

- 5.7 As the temperature varies, the frequency  $f$  of a tuning fork is related to its linear dimensions  $L$  and the value of the Young modulus  $E$  of its material by

$$f \propto \sqrt{EL}.$$

When the temperature rises by 10 K, the frequency of a certain fork falls by  $(0.250 \pm 0.002)\%$ . For the same temperature rise, the Young modulus of the material falls by  $(0.520 \pm 0.003)\%$ . Calculate the value of  $\alpha$ , the linear expansivity of the material, given by these experiments. What is its standard error? Is this a good method for measuring the linear expansivity?



## Part 2

### Experimental methods



### 6.1 Introduction

In this chapter we shall consider some general principles for making measurements. These are principles which should be borne in mind, first in selecting a particular method and second in getting the most out of it. By the latter we mean making the method as precise or reproducible as possible, and – even more important – avoiding its inherent systematic errors.

We shall illustrate the various points by describing some specific examples of instruments and methods. Though chosen from several branches of physics, they are neither systematic nor exhaustive. The idea is that, having seen how the principles apply in these cases, you will be able to apply them yourselves in other situations. As always there is no substitute for laboratory experience. But experience without thought is a slow and painful way of learning. By concentrating your attention on certain aspects of measurement making we hope to make the experience more profitable.

### 6.2 Metre rule

We start with almost the simplest measuring device there is – a metre rule. Its advantages are that it is cheap to make and convenient to use. It can give results accurate to about  $\frac{1}{2}$  mm. However, to achieve this accuracy certain errors must be avoided.

(a) *Parallax error.* If there is a gap between the object being measured and the scale, and the line of sight is not at right angles to the scale, the reading obtained is incorrect (Fig. 6.1a). This is known as a *parallax error* and clearly may occur, not only in a metre rule, but in all instruments where a pointer moves over a scale. It may be reduced by having the object or pointer as close to the scale as possible, and also by having a mirror next to the scale as shown in Fig. 6.1b. Aligning the image of the eye with the object ensures that the line of sight is at right angles to the scale.

(b) *Zero error.* Except for crude measurements it is not good practice to place one end of the rule against one end of the object and take the reading at the other end (Fig. 6.2a). Instead, the object should be placed so that a reading can be taken at

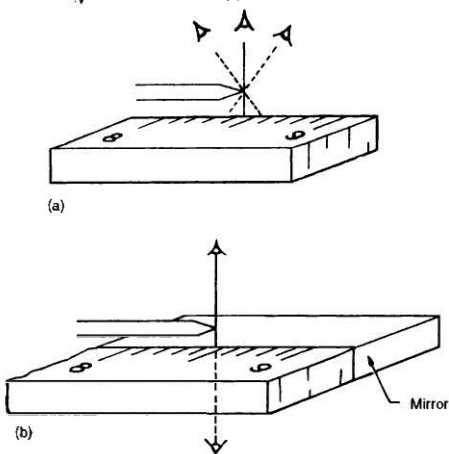


Fig. 6.1. Parallax error. (a) Different positions of the eye result in different readings. (b) A mirror placed beside the scale ensures that the line of sight is at right angles to the scale.

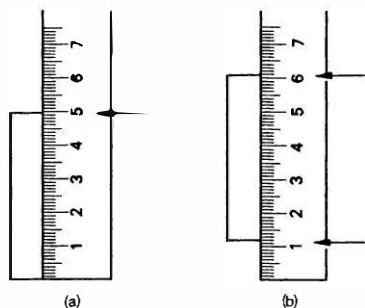


Fig. 6.2. Measuring the length of an object as in (a) is bad practice. It will give a systematic error if the end of the rule is worn. The rule should be placed as in (b) and two readings taken.

both its ends (Fig. 6.2b). This is because the end of the rule may be worn or the zero ruling incorrect in some way. In general the zero position of any instrument should be regarded as suspect. The resulting error can usually be avoided by a simple subtraction technique as here.

(c) **Calibration.** The scale on the rule may be incorrectly marked. The rule should therefore be checked, or calibrated. This is done by simply laying it beside a more accurate standard rule and noting the readings.

It is important to realize the logic of this. The ordinary metre rule is cheap because it is made of a cheap material – wood – and the scale is engraved without a great deal of care. The two factors go together; it is not worth engraving an accurate scale on a wooden rule whose whole length is liable to change with time.

Suppose we have say 20 experimenters in a laboratory, and they need to measure lengths of about 500 mm to an accuracy of  $\frac{1}{10}$  mm. We could supply each one with a steel rule known to be marked to this accuracy. Such a rule is far more expensive than an ordinary wooden one, whose markings are probably only good to about  $\frac{1}{2}$  mm over the whole rule. (Plastic rules are much worse than this and are often in error by as much as 1%.) Alternatively, we could supply each person with an ordinary rule and install *one* expensive standard in the laboratory. The cost of this would only be a fraction of the other. But the measurements could still be made to the required accuracy, *provided each experimenter remembers to do the calibration.* This procedure, viz.

many cheap + one expensive standard + calibration

is obviously a sensible one.

If you use only part of the rule during a particular experiment, the comparison with the standard should be done particularly carefully for the part actually used. It would not matter if the rest was incorrectly marked. In practice it is unlikely that one part of a metre rule is ruled less accurately than another. But it is a good general principle when calibrating an instrument to concentrate on the range actually used.

### 6.3 Micrometer screw gauge

This instrument measures the external dimensions of objects up to the order of 100 mm. A typical micrometer is shown in Fig. 6.3. The spindle has two threads to the millimetre, so that one complete rotation of the thimble T corresponds to 500  $\mu$ m. The instrument can easily be read to 10  $\mu$ m, a gain in precision of about 20 over the metre rule. High precision versions of the instrument can be read to 1  $\mu$ m.

Several points may be noted:

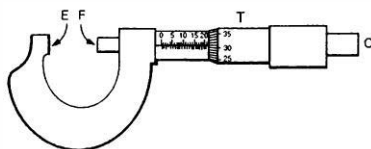


Fig. 6.3. Micrometer screw gauge.

(a) The increase in precision comes from the screw mechanism – a highly effective way of magnifying linear motion.

(b) The personal element involved in the torque applied to the thimble, when the face  $F$  is in contact with the object, is eliminated by a ratchet device. The thimble is rotated by applying torque, not to the body itself, but to the end cap  $C$ . This transmits the motion only up to a certain standard torque; thereafter it slips, and rotating it further has no effect on the reading. So the final reading corresponds to a standard pressure of the face  $F$  against the object.

(c) The instrument is prone to zero error, which should always be checked by taking a reading with the face  $F$  right up against  $E$ .

(d) Other readings may be checked by means of gauge blocks, which are rectangular blocks of high-grade hardened steel. The end-faces of a block are flat and parallel, and the distance between them, known to about  $0.1\ \mu\text{m}$ , is engraved on the block.

## 6.4 Measurement of length – choice of method

In the last two sections we have considered two methods of measuring length. We are not going to describe any more in detail, but, instead, shall look at the general problem of measuring length.

First we must decide what we mean by length. On the whole we know fairly well what we mean in the case of objects whose velocity is small compared with that of light, and which we can see, or almost see, in the laboratory or near the Earth and Sun – a range in length say of about  $10^{-8}$  to about  $10^{11}$  m. But when we consider objects whose velocity is not small compared with that of light, or try to extend the range downwards to objects as small as atoms and elementary particles, or upwards to the distances that separate us from the stars and galaxies, we have to say what we mean by length and distance – we cannot lay a ruler across a nucleus or extend one to a star. We have to say something like ‘if we do such and such an experiment, the result is as though something extends over such and such a distance, or is a certain distance away’. Such *operational* definitions sometimes lead to concepts of length and distance which differ from our usual common-sense ideas on the subject. But the latter have come from seeing and thinking about lengths over a very restricted range. So we need not be too surprised if they do not apply when the range is enormously extended.

The problems of definition and measurement at the extreme ends of the range are outside the scope of this book. We shall only consider the range where our common-sense ideas do apply. However, it does no harm to put the discussion in a more general context and to remind ourselves that behind *all* measurements lies the definition – usually implicit – of the quantity being measured.

Even restricting ourselves to measurements of length of the order of  $1\ \mu\text{m}$  to  $1\ \text{m}$ , we have a range of instruments to choose from. In deciding which one to use we should consider the following:

Table 6.1. *Suitability of some instruments for measuring length*

Instrument	Range/m	Precision/ $\mu\text{m}$	Applicability
metre rule	1	200	general
vernier calipers	0.1	50	most useful for overall dimensions of an object, also for dimensions of a space – width of gap, diameter of hole
cathetometer	1	10	general
micrometer screw gauge	0.1	2	overall dimensions of an object
travelling microscope	0.2	1	general

(a) The nature of the length we wish to measure, for example, whether it is the distance between two marks or between the two end-faces of a rod or bar, or the diameter of a hole or of a rod.

(b) The rough value of the length.

(c) The precision required.

In Table 6.1 we list five instruments which are available in the range we are considering. Vernier calipers are shown in Fig. 6.4. A cathetometer is simply a telescope that can slide on a rigid graduated bar. A travelling microscope is an instrument in which the distance travelled by the microscope, from a position in which one object or mark is in focus on its crosswires to another similar position, is given by the rotation of a screw device that controls the motion. (This instrument is subject to the error known as ‘backlash’. Owing to looseness in the moving parts, the reading depends on the direction from which the crosswires are moved into position. The error is avoided if the final setting is always made from the same direction.) The table shows the approximate range and precision, and type of measurement for each instrument.

Of course, having chosen the right instrument we have to make the right set of measurements with it. In theoretical physics we say a cylindrical rod has a diameter of  $d$  mm and that is the end of the matter. But in practical physics we have to verify that the rod is a cylinder – or, to be more correct, to say within what limits it is. So at one place along the length we should measure the diameter in various directions. Then we should repeat the measurements at various places along the length. As always, the thoroughness of the investigation depends on the purpose of the measurements. Similarly, before giving a distance between the end faces of a bar, we must check the extent to which the faces are parallel.

We mention briefly how the range of lengths and the precision of the measurements may be increased. The latter may be achieved by optical interference experiments. Similar experiments with various forms of waves – light, X-rays, electrons, neutrons – also extend the range in the downward direction. Another way of measuring changes in length is to measure the change in capacitance when one plate of a capacitor moves relative to another (Sydenham 1985). At the upper

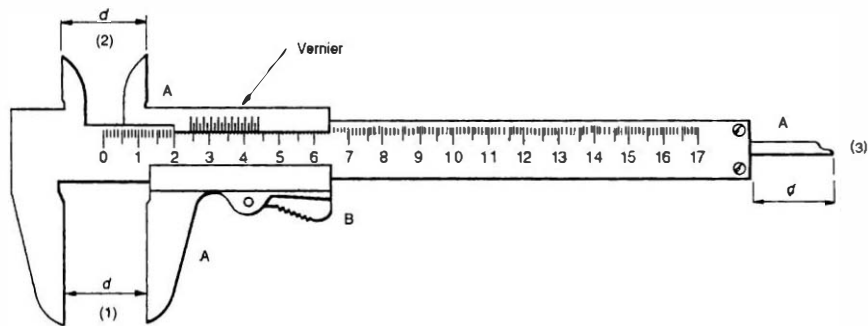


Fig. 6.4. Vernier calipers. The parts marked A form a rigid unit, which is free to move relative to the rest of the instrument when the spring-loaded button B is pressed. The three distances marked  $d$  are equal and are read off from the vernier scale. (1) gives the diameter of a rod, (2) the diameter of a hole, and (3) the depth of a blind hole.



end of the length range we may measure distances by triangulation. The enormous cosmological distances are deduced – indirectly and with varying degrees of plausibility – from measurements of quantities such as the apparent brightness of stars (Pasachoff *et al.* 1994).

All is grist to the physicist's mill. We look for any physical phenomenon that depends on the quantity we wish to measure. From the variety of instruments based on these phenomena, we select one of appropriate applicability, range, and precision.

### 6.5 Measurement of length – temperature effect

In any precise measurement of length we must consider the effects of thermal expansion. This applies first to the object being measured. Suppose, for example, we require the length of a tube at liquid-hydrogen temperature (20 K). If we measured its length at room temperature and did not allow for the contraction, we should make a serious error. In this extreme case, to calculate the correction we could not assume that the linear expansivity was independent of temperature, but would have to know its values from 300 K down to 20 K.

Second, we must consider whether thermal expansion is affecting the readings of the instrument used to measure the lengths. And last, if we calibrate the instrument by measuring the length of a standard object such as a gauge block, we must consider whether the length of the latter has departed from its nominal value owing to temperature change.

In Table 6.2 the values of  $\alpha$ , the linear expansivity, for a few common substances are given. The values are approximate and correspond to room temperature (293 K). Invar is a steel alloy containing 36% nickel; its chief merit is its very low expansivity at room temperature, and it is often used when this property is required. At higher temperatures its value of  $\alpha$  increases. Fused silica is superior as a low-expansion substance; the value of  $\alpha$  given in the table applies up to 1300 K. Moreover, it is extremely stable dimensionally. It is therefore often

Table 6.2. *Linear expansivity  $\alpha$  for some common substances*

Substance	$\alpha/10^{-6} \text{ K}^{-1}$
copper	17
brass	19
steel	11
Invar	1
soda glass	9
Pyrex	3
fused silica	0.5
wood – along grain	4
across grain	50

used in precise measurements where the geometry must be very exactly defined and must vary as little as possible with temperature.

Let us look at the actual values in the table. The substances listed are fairly representative. Most solids at room temperature have values of  $\alpha$  in the range  $5$  to  $25 \times 10^{-6} \text{ K}^{-1}$ . If we take  $10^{-5} \text{ K}^{-1}$  as a typical value, we see that a change in temperature of  $10 \text{ K}$  corresponds to a change in length of  $1$  part in  $10^4$ .

It is very important in practical physics to have an idea of orders of magnitude of this kind. On the one hand, if we do not correct for thermal changes in dimensions in a precise measurement, we may make a serious error. On the other hand, if we were to calculate the possible thermal correction every time we measured a length of any kind, we would obviously waste a lot of time. Common sense tells us this. But what common sense does not tell us is at what stage, as the precision of an experiment is increased, we should start to concern ourselves with the effect. This is where the order of magnitude calculation comes in. It tells us that, for temperature changes of less than  $10 \text{ K}$ , we need not worry about temperature effects in length, unless we are measuring it to better than  $1$  part in  $10^4$ .

## 6.6 The beat method of measuring frequency

(a) *The phenomenon of beats.* Suppose we have two sinusoidal waves of equal amplitude  $A$  but slightly different frequencies  $f_1$  and  $f_2$ . We do not specify the physical nature of the waves. They may, for example, be the displacement in space of a mechanical vibrating system or the voltage across a capacitor in an oscillatory circuit. Whatever their nature we can represent the separate displacements by

$$y_1 = A \cos 2\pi f_1 t \quad \text{and} \quad y_2 = A \cos 2\pi f_2 t. \quad (6.1)$$

They are shown in Fig. 6.5a and b. If the two waves act together, the total displacement is

$$\begin{aligned} y &= y_1 + y_2 = A(\cos 2\pi f_1 t + \cos 2\pi f_2 t) \\ &= 2A \cos 2\pi \frac{f_1 - f_2}{2} t \cos 2\pi \frac{f_1 + f_2}{2} t. \end{aligned} \quad (6.2)$$

Suppose  $f_1$  and  $f_2$  are fairly close in value, that is,

$$f_1 - f_2 \ll f_1 + f_2. \quad (6.3)$$

Then we may regard the factor  $\cos 2\pi[(f_1 + f_2)/2]t$  in (6.2) as representing rapid sinusoidal motion, the amplitude of which is varying slowly according to the factor  $2A \cos 2\pi[(f_1 - f_2)/2]t$ . The overall motion is shown in Fig. 6.5c.

Quite apart from (6.2), derived from the previous line by straightforward trigonometry, we can see how the waves in (a) and (b) in Fig. 6.5 add to give the wave in (c). At time P the two waves are in phase, and their sum is large. As their frequencies are slightly different, they gradually get out of phase until at Q they

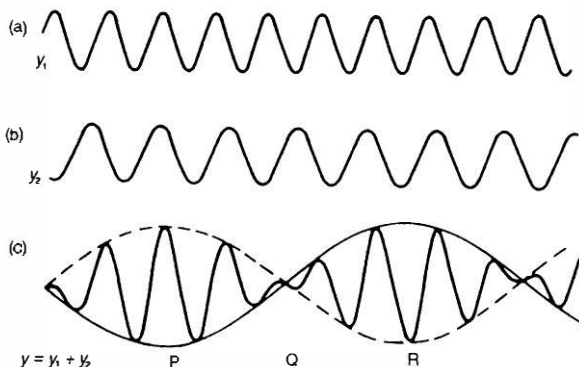


Fig. 6.5. The two waves  $y_1$  and  $y_2$ , of slightly different frequency, add to give the resultant shown in (c).

are exactly  $180^\circ$  out of phase, and their sum is zero. At R they are back in phase again. This phenomenon, the successive swelling up and dying down of the resultant, is known as *beats*, and the frequency of the maxima in the amplitude variation is known as the *beat frequency*.

The broken line in Fig. 6.5c has a frequency  $(f_1 - f_2)/2$ . But the maximum amplitude occurs twice in each of its cycles. So the frequency of the beats is given by

$$f_b = f_1 - f_2. \quad (6.4)$$

This is an important result.

**(b) Measurement of frequency.** The phenomenon of beats provides a very precise method of measuring the frequency  $f$  of a source, particularly for electromagnetic waves. For such waves we can produce a standard source whose frequency  $f_0$  is known very precisely indeed – see section 7.4. If we mix its output with that of the unknown and measure the beat frequency  $f_b$ , we have

$$f = f_0 \pm f_b. \quad (6.5)$$

The precision of the method lies in the fact that we can find a standard oscillator such that  $f_0$  is close to  $f$ . Then  $f_b$  is a small quantity; so even a relatively imprecise measurement of it suffices to determine  $f$  precisely. This is an example of the Case I situation mentioned on p. 48.

Suppose, for example,

$$f_0 = 1\,000\,000 \text{ Hz} \quad \text{and} \quad f_b = 500 \pm 5 \text{ Hz}.$$

Then

$$f = 1\,000\,500 \pm 5 \text{ Hz.} \quad (6.6)$$

In other words, a measurement of  $f_b$  to 1 part in 100 has given the value of  $f$  to 1 part in 200000.

To measure  $f_b$  it is necessary first to extract from  $y$  in Fig. 6.5c a signal equivalent to the positive envelope of the wave, a process known as *detection* or *demodulation* (Horowitz and Hill 1989). The frequency of the signal is then measured by some standard method, e.g. by converting the signal into a series of pulses, one for each cycle, and counting the number of pulses in a known time interval.

We have taken the positive sign – arbitrarily – in (6.5) to obtain the value of  $f$  in (6.6). How would we know which sign to take in practice? One way would be to measure  $f$  approximately to find out whether it was greater or less than  $f_0$ . In the present example a measurement precise to slightly better than 1 part in 2000 would be adequate for the purpose. Another way of selecting the correct sign is to make a slight change in  $f_0$ , say to increase it slightly, and observe whether  $f_b$  increases or decreases.

## 6.7 Negative feedback amplifier

(a) *Principle of negative feedback.* Suppose we have an electronic amplifier which has the property that, when a voltage  $V_i$  is applied across its input terminals, a voltage  $V_o$  appears across its output terminals, where

$$V_o = \alpha V_i. \quad (6.7)$$

We assume  $\alpha$  is a constant and call it the *intrinsic gain* of the amplifier. We shall not concern ourselves with the internal details of the amplifier but shall simply represent it as a box with input and output terminals (Fig. 6.6).

Suppose now that we have a certain signal voltage  $V_s$ , that we wish to amplify, but that, instead of applying  $V_s$  directly to the input terminals, we subtract from it

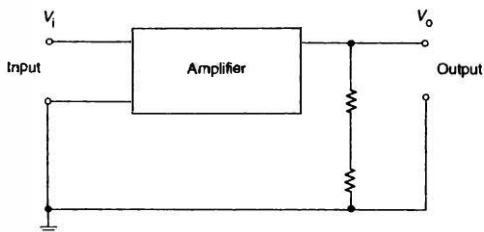


Fig. 6.6. Schematic representation of amplifier.

a fraction  $\beta V_o$  of the output and apply the remainder to the input terminals. This procedure of reducing the input signal by something that depends on the output is known as *negative feedback*. Figure 6.7 is a schematic representation of the situation where the feedback signal is obtained from a simple resistance chain.

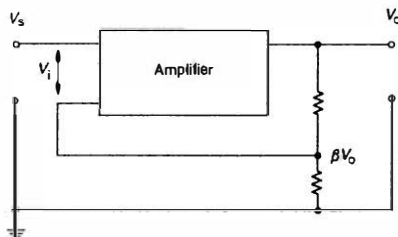


Fig. 6.7. Amplifier with negative feedback.

We have

$$V_i = V_s - \beta V_o. \quad (6.8)$$

Therefore

$$V_o = \alpha V_i = \alpha(V_s - \beta V_o), \quad (6.9)$$

whence

$$\frac{V_o}{V_s} = \frac{\alpha}{1 + \alpha\beta}. \quad (6.10)$$

So the net of overall gain is reduced by the feedback, since  $\alpha$  and  $\beta$  are both positive numbers.

Now what is the point of this? The answer comes from considering what happens when  $\alpha\beta$  is very much larger than unity. In that case we may neglect the 1 in the denominator of (6.10), and the gain becomes  $1/\beta$ . In other words, the net gain does not depend on the intrinsic gain of the amplifier, but only on the fraction  $\beta$  of the output that is fed back.

**(b) Advantages of the method.** (i) *Insensitivity to variations in supply voltage and amplifier components.* The quantity  $\beta$  can be fixed very precisely. It simply depends on having a pair of well-defined resistors. The quantity  $\alpha$  on the other hand can vary for many reasons. For example, it may fluctuate owing to changes in the supply voltage. Again, it may vary from one amplifier to another owing to small changes in the various components – resistors, capacitors, and transistors. Nevertheless, provided the value of  $\beta$  is the same for all the amplifiers, their

overall gains will be almost equal. The fact that negative feedback reduces the gain of an amplifier is not really a disadvantage, because pure gain as such is quite easy to achieve.

Quite apart from (6.10), it is easy to see *qualitatively* why negative feedback reduces the sensitivity of the net gain to the intrinsic gain of the amplifier. Suppose the amplifier is working with feedback at a certain value of  $\alpha$ ,  $\beta$ , and  $V_s$ . Then  $V_o$  and  $V_i$  are given by (6.10) and (6.8). Now suppose that for some reason the value of  $\alpha$  is reduced. This tends to reduce  $V_o$ , which reduces the feedback voltage  $\beta V_o$ , which results in an *increase* in  $V_i$ . Therefore the output  $V_o$  is not reduced as much as it would otherwise be. Of course it must be reduced by something, however small; otherwise the whole sequence of steps that we have just outlined would not occur at all.

Let us put in some numbers to show how negative feedback reduces the effect of a change in  $\alpha$  on the net gain of an amplifier. If

$$\alpha = 20000 \quad \text{and} \quad \beta = \frac{1}{100}, \quad (6.11)$$

then

$$\text{net gain} = \frac{20000}{1 + 200} = 99.50. \quad (6.12)$$

Now suppose that  $\alpha$  drops to 10000 – a very large drop indeed. The net gain becomes

$$\frac{10000}{1 + 100} = 99.01. \quad (6.13)$$

So though the intrinsic gain changes by a factor of 2, the net gain changes by only  $\frac{1}{2}\%$ .

(ii) *Improved frequency response.* Suppose the input is a sinusoidal voltage. For most amplifiers the intrinsic gain depends on frequency, owing to the various capacitances in the circuit. The advantage of obtaining the feedback voltage from a resistance chain is that  $\beta$  does not depend on frequency. So the overall gain of the amplifier is practically independent of frequency.

In a hi-fi audio amplifier, heavy negative feedback is used to achieve this situation, which means that musical notes of all frequencies, and their harmonics, are amplified by the same amount. This is necessary for the final result to be a faithful copy of the original, and the specification of the amplifier usually contains a statement about the constancy of the overall gain over a specified frequency range.

(iii) *Improved linearity.* If, for a signal of given frequency,  $V_o$  is not a linear function of  $V_i$ , we still write

$$V_o = \alpha V_i, \quad (6.14)$$

but  $\alpha$  now varies with  $V_i$ . However, provided  $\alpha\beta \gg 1$  for all values of  $V_i$ , the same reasoning applies as before, and

$$V_o \approx \frac{V_s}{\beta} \quad (6.15)$$

So, provided  $\beta$  is constant and  $\alpha\beta \gg 1$ ,  $V_o$  is approximately a linear function of  $V_s$ , and this is true no matter how non-linear the relation between  $V_o$  and  $V_i$ .

Other advantages of negative feedback are that the input impedance of the amplifier is increased and its output impedance decreased.

(c) **Stability.** If the sign of the feedback signal is reversed, so that it *adds* to the signal  $V_s$  instead of subtracting from it, we have *positive* feedback. Equation (6.10) still applies with  $\beta$  negative. It is now possible for  $\alpha\beta$  to be equal to  $-1$ , in which case the equation gives an infinite value for  $V_o$ . What happens in practice is that  $V_o$  rises to a large value, the capacitive and inductive elements in the circuit give time delays, and the system becomes unstable and starts to oscillate. Circuits based on this principle are in fact designed deliberately as oscillators.

However, if the circuit is required as an amplifier, such behaviour is highly undesirable. The danger with any type of feedback circuit is that even though we intend always to have negative feedback, we may in fact get positive feedback at certain frequencies. This is because the amplifier produces phase shifts in the signals and these shifts depend on frequency. Part of the art of designing an amplifier is to ensure that it remains stable at all frequencies. The theoretical conditions for this have been worked out by *Nyquist and others*.

For a full discussion of the theory and design of negative feedback amplifiers you are referred to one of the books listed on p. 206. In this section we have given only an introduction to this important idea, which may be applied to all amplifying devices.

## 6.8 Servo systems

(a) **The servo principle.** Suppose we have an apparatus S, some feature of which we wish to control from a unit C. We may take a control signal from C, put it through an amplifier A and apply the output to S.

Suppose now we allow S to produce a signal F that is a measure of the quantity we are trying to control, feed it back so as to subtract from the control signal and apply the difference to the input terminals of the amplifier – Fig. 6.8. This system of controlling the apparatus is known as a *servo* system. We see that it is based on the idea of negative feedback, and many of the advantages of negative feedback that we considered in the last section in relation to amplifiers apply also to quite general servo systems.

(b) **Example – temperature control.** As an example of the servo principle we consider a bath whose temperature  $T$  we require to keep constant, at a value that depends on the setting of a control. The bath – Fig. 6.9 – loses heat to its surroundings, and its temperature is maintained by means of a heating coil H. Let the control produce a voltage  $V_c$ . The feedback is provided by some device that

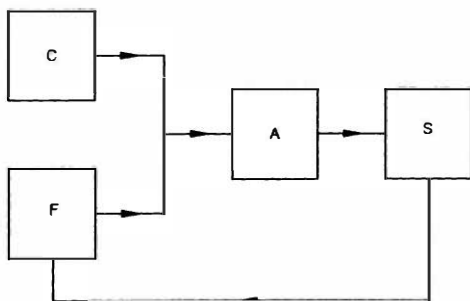


Fig. 6.8. Block diagram of servo system.

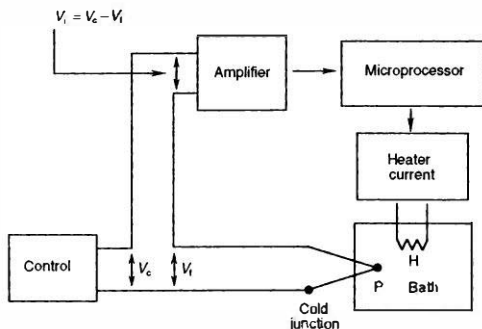


Fig. 6.9. Servo system for maintaining bath at constant temperature.

gives a signal that depends on the temperature of the bath, for example a thermocouple with its hot junction at P. Suppose this produces a voltage  $V_f$ . The voltage  $V_c - V_f$  is fed into an amplifier, the output of which goes to a microprocessor. The output from this unit controls the current through the heating coil.

The required temperature  $T$  is that for which the thermocouple output  $V_f$  is equal to  $V_c$ . When the bath is at this temperature, the input signal to the amplifier is zero. The microprocessor is programmed so that under this condition the heat supplied to the bath is constant. At equilibrium the heat supplied is equal to the heat loss of the bath at temperature  $T$ . If for any reason the temperature should fall, for example because the heat loss increases due to a fall in the surrounding temperature,  $V_f$  falls, and the positive signal  $V_c - V_f$  acts via the amplifier, microprocessor, and heating unit to increase the current through the heater.



Conversely, if the temperature rises above  $T$ ,  $V_c - V_f$  becomes negative, and this results in a decrease in the heating current.

We see then the essential features of a servo system. We must have a control signal  $V_c$ . This acts as a reference. We must have an output signal  $V_f$ , which is a measure of the physical quantity that is being controlled. The difference between the two signals is used to operate a device which acts so as to reduce this difference to zero. In short a servo system provides *automatic control*.

(c) **Stability.** So far we have ignored time lags in the system. However, these have an important effect on its behaviour. Suppose in the example we move the control to correspond to a new higher temperature  $T'$ . While the bath is warming up and  $T$  is less than  $T'$ , the servo mechanism is providing higher and higher currents to the heater. This continues until the thermocouple reaches  $T'$ . However, owing to thermal time lags in the system, heat continues to come in at a rate greater than the equilibrium heat loss at temperature  $T'$ . So the temperature of the bath rises above this value. Whereupon the servo mechanism starts to reduce the heating current, and again, owing to thermal time lags, it reduces it too much.

This type of oscillatory behaviour in a servo system is known as *hunting*. It is analogous to the instability we mentioned previously in connection with positive feedback in amplifiers. Mathematically the two types of behaviour are equivalent. Instability in servo systems is overcome by damping, that is, by diminishing the change in the applied quantity – heat into the bath in the present example – in various ways. The microprocessor may be programmed to respond, not only to the instantaneous temperature of the bath, but also to its past history, so that a time-varying heater current is applied such that the bath is brought to the required temperature  $T'$  as quickly as possible, with minimum, or indeed zero, overshoot.

## 6.9 Natural limits of measurement

It might be thought that, if we used sufficiently sensitive instruments and took enough care, we could make measurements as precisely as we pleased. But this is not so. Quite apart from the limitations imposed by the uncertainty principle in quantum physics, with which we are not here concerned, there are several phenomena which give rise to random fluctuations in measuring devices. These fluctuations are known as *noise* and provide natural limits to the precision that can be achieved.

(a) **Brownian motion.** One source of random fluctuation in a measuring instrument is Brownian motion; an example is provided by a small mirror suspended by a torsion fibre. If the restoring couple due to an angular displacement  $\theta$  is  $c\theta$ , then the potential energy is

$$V = \frac{1}{2} c \theta^2. \quad (6.16)$$

The mirror is being constantly bombarded by gas molecules, and, though the value

of the couple they exert is zero when averaged over time, its instantaneous value is not, and the mirror undergoes random fluctuations about its mean position.

The mirror is in thermal equilibrium with the gas molecules. So, by the theorem of the equipartition of energy, the average value of  $V$  is  $\frac{1}{2}kT$ , where  $k$  is the Boltzmann constant and  $T$  the thermodynamic temperature.\* The average value of  $\theta^2$  is therefore

$$\overline{\theta^2} = \frac{kT}{c}. \quad (6.17)$$

Since  $k = 1.38 \times 10^{-23} \text{ J K}^{-1}$ , the effect is usually small at room temperature. However, for small values of  $c$ , the fluctuations become appreciable; in fact, the phenomenon has been used to provide a value for  $k$  (Kappler 1938).

Notice that the effect cannot be reduced by reducing the pressure. Such a reduction decreases the number of molecules striking the mirror per second and changes the motion from a high speed jitter to one with sinusoidal swings, but the average value of  $\theta^2$  remains the same. Some diagrams of the motion are given in Fowler 1936, p. 783.

**(b) Johnson noise.** A second and important type of thermal agitation is the phenomenon known as *Johnson* or *thermal noise*. A resistance  $R$  at temperature  $T$  is found to act as a generator of random emfs. They may be regarded as arising from the thermal motion of the conduction electrons. The resulting emf  $E$  varies in an irregular manner, and, if a frequency analysis is made, the average value of  $E^2$  in the frequency range  $f$  to  $f + df$  is

$$\overline{E^2} = 4RkT df. \quad (6.18)$$

We are simply quoting this result.† It follows from basic thermodynamic reasoning and is valid provided  $f \ll kT/h$ , where  $h$  is the Planck constant. (At room temperature  $kT/h = 6 \text{ THz}$ .)

Notice that the right-hand side of (6.18) does not contain  $f$  itself. In other words, except at very high frequencies, Johnson noise is constant throughout the frequency spectrum and is proportional to the product  $RT$ . For this reason, when the signal to be amplified is very weak and the noise therefore very troublesome, the amplifier is sometimes cooled to liquid-nitrogen and even liquid-helium temperatures. This not only reduces the value of  $T$  but also that of  $R$ .

**(c) Noise due to the discreteness of matter.** Electric current is carried by discrete particles – electrons and electron holes. In successive intervals of time the number of particles in motion fluctuates, a phenomenon known as *shot noise*. The smaller

\* The equipartition theorem is discussed in most textbooks on statistical mechanics; see for example Zemansky and Dittman 1997.

† For comprehensive accounts of noise in measurements see Robinson 1974 and Milburn and Sun 1998. The former gives a proof of (6.18). For a good introduction to frequency (Fourier) analysis see Wilson 1995.

the current, the larger the fractional fluctuation. Examples of shot noise are the fluctuations in the current of a semiconductor diode due to variations in the rate at which the charge carriers cross the p-n junction, and fluctuations in electron emission by a thermionic cathode or photocathode.

(d) **Flicker noise.** In addition to Johnson noise and shot noise, which are fundamental and do not depend on the quality of the components in which they occur, there is another type of noise which depends on the detailed properties of the component. It is known as *flicker* or  $1/f$  noise because its frequency spectrum varies as  $1/f$ . Thus it is most important at low frequencies. The physical origin of flicker noise varies from one device to another. It commonly arises from variations of properties with time in devices which are not in thermal equilibrium, but are subject to an external disturbance such as a bias or signal voltage. For example, a carbon resistor consists of a large number of small granules with contact resistance between them. When a current passes, small random motions of the granules produce changes in the overall resistance, and the voltage across the resistor fluctuates with time. In a biased semiconductor device, flicker noise arises from fluctuations in the rates at which the majority and minority carriers are generated and recombine in heterogeneous regions of the crystal.

(e) **Noise in general.** Except for work involving the detection of very weak signals, the various sources of noise we have been describing are not usually limiting factors in normal laboratory measurements. Other sorts of disturbance are often present and may be more serious. Common examples are pickup from the mains, interference by nearby electrical machinery, spurious signals due to bad electrical contacts and faulty electronic components.

## Exercises

6.1 A stroboscope is a device for measuring the frequency  $f$  of a rotating object by viewing it with a flashing light of known frequency  $f_0$  and measuring the apparent frequency  $f_{app}$  of rotation.

(a) Show that if  $f'$  is roughly equal to  $mf_0$ , where  $m$  is an integer,

$$f_{app} = f - mf_0.$$

What is the significance of a negative value for  $f_{app}$ ?

(b) If

$$m = 5,$$

$$f_0 = 100.00 \pm 0.01 \text{ Hz},$$

$$f_{app} = 0.40 \pm 0.05 \text{ Hz},$$

calculate the value of  $f$  and its standard error.

6.2 The period  $T$  of a rigid pendulum is determined by measuring the time  $t$  taken for an integral number of swings  $N$ . The error  $\Delta t$  in  $t$  comes from starting and stopping the timer and may be assumed to be independent of  $t$ . So the larger the value of  $N$  and hence of  $t$ , the more precise is the value of the period.

Let the value of  $\Delta t$  be 0.2 s. Twenty swings are counted and are found to take a time  $t = 40.8$  s. The pendulum is set swinging again; this time the swings are not counted, but an integral number  $N_1$  are found to take 162.9 s. It is set swinging a third time, and  $N_2$  swings are found to take 653.6 s. Deduce the values of  $N_1$  and  $N_2$ , and the final value of  $T$  with its error. (Assume that the amplitude of the swings is sufficiently small for the variation in the period to be negligible throughout the measurements.)

- 6.3 Make a critical comparison of the following types of thermometers on the basis of range of temperature, precision, convenience (including situations of particular applicability), and cost:
- mercury in glass,
  - thermocouple,
  - platinum resistance,
  - thermistor,
  - constant volume gas thermometer,
  - optical pyrometer.

- 6.4 Make a list of possible ways of measuring magnetic fields and compare them as in the previous exercise.

The following discussion exercises are meant to make you think about the nature of measurement and to help you see how measurement and theory are inextricably bound together. Some of them involve ideas beyond the scope of the average first-year undergraduate course.

- 6.5 Discuss the concept of size as applied to
- an atom,
  - an atomic nucleus.
- 6.6 Discuss appropriate methods of measuring the distance between
- atoms in a crystal,
  - the two atoms in a hydrogen molecule,
  - two points about 10 km apart on the Earth's surface,
  - the Earth and the Moon,
  - the Earth and a nearby star,
  - the Earth and a distant star.
- 6.7 Discuss appropriate methods of measuring the mass of
- a sack of potatoes,
  - a bar of gold,
  - a proton,
  - a neutron,
  - the Earth.
- 6.8 Explain what is meant by the following statements and how they may be verified:
- The temperature of a salt following adiabatic demagnetization is 0.001 K.
  - The temperature in a plasma is 50 000 K.
  - The temperature in outer space is 3 K.

In the present chapter we consider some examples of experimental techniques. They have been chosen because they contain many ingenious features and show that the same principles of good experimentation apply whether the experiment be advanced or elementary. They come from different branches of physics – optics, electricity, mechanics, and atomic physics – and are illustrated in different contexts – either as an instrument, a complete experiment, or an application. This is the hardest chapter in the book, as some of the physics may be unfamiliar to you. But each section stands alone, so you can omit any one of them – or indeed the whole chapter – at first reading, and carry on with the rest of the book. On the other hand, the experimental ideas contain so many instructive features that you will find a little perseverance well rewarded.

### 7.1 Rayleigh refractometer

(a) *Description of instrument.* The Rayleigh refractometer is an instrument devised to measure the refractive indices of gases and also small changes in the refractive indices of solids and liquids.

Monochromatic light from a vertical slit  $S$  (Fig. 7.1) is collimated by an achromatic lens  $L_1$  and falls on two vertical slits  $S_1$  and  $S_2$ . The two beams pass through tubes  $T_1$  and  $T_2$  of equal length  $l$ , lying in the same horizontal plane. The beams then recombine to form vertical interference fringes in the focal plane of the lens  $L_2$ . The fringes are viewed by a small cylindrical lens  $L_3$ .

To measure the refractive index of a gas, the fringes are first observed with both tubes evacuated. Gas is then admitted to one of the tubes, say  $T_1$ , thereby increasing the optical path of beam 1. This causes the fringe pattern to move

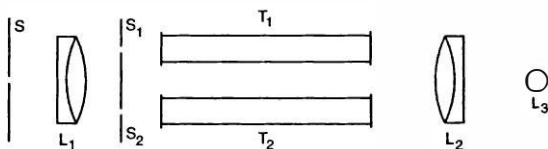


Fig. 7.1. Rayleigh refractometer – view from above.

sideways and the number of fringes passing a crosswire is counted. If  $p$  (not necessarily an integer) fringes pass, the refractive index  $\mu$  of the gas is given by

$$t(\mu - 1) = p\lambda, \quad (7.1)$$

where  $\lambda$  is the wavelength of the light. We consider some practical aspects of the instrument in the following sections; a detailed account of the Rayleigh refractometer will be found in Ditchburn 1952.

**(b) Reference system.** Instead of using a crosswire as the reference or fiduciary mark, it is better to use a second set of fringes which remain fixed. These are obtained by allowing only the upper halves of the beams from  $S_1$  and  $S_2$  to pass through the tubes. The lower halves pass under them and produce a second, independent set of fringes. Since the optical paths of the two lower beams are identical, the second set of fringes remains fixed throughout the experiment.

The advantage of this type of reference mark is that the eye is much more sensitive to relative displacements of two similar non-overlapping parallel lines, in this case the two sets of fringes, than to relative displacements between a crosswire and a fringe (see Fig. 7.2). In the first case, displacements of as little as  $\frac{1}{40}$  of a fringe separation can be detected; whereas in the second, the limit is about  $\frac{1}{10}$  of a fringe separation. The sensitivity of the eye in the first case is known as *vernier acuity*.

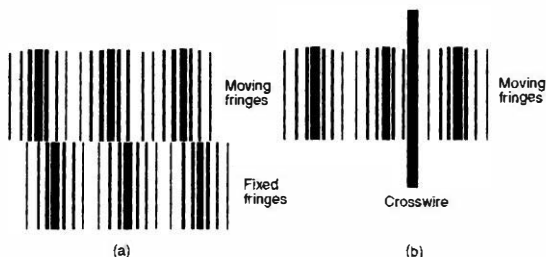


Fig. 7.2. Two forms of reference marks. (a) fixed set of fringes, (b) crosswire. The eye detects small movements more readily in (a) than in (b).

Another important advantage of the fringe reference system is that distortion of the framework of the apparatus or displacement of the double slit  $S_1S_2$  does not affect the readings, because these faults affect both sets of fringes in the same way.

**(c) Cylindrical eyepiece.** In a practical case the separation  $s$  of the slits  $S_1$  and  $S_2$  is about 10 mm. The angular separation of the fringes is given by

$$s\theta = \lambda. \quad (7.2)$$

So for  $\lambda = 500 \text{ nm}$ ,  $\theta$  is  $5 \times 10^{-5} \text{ rad}$  or about  $\frac{1'}{5}$ . The fringes are thus very close together. They are viewed with the lens  $L_3$ , which is simply an accurately made cylindrical glass rod of diameter about 2 mm. The point of having a vertical cylindrical lens is that it only gives magnification in the direction of the separation of the fringes, which is where we want it. The effective aperture of the beam is much smaller than the pupil of the eye; any magnification therefore reduces the brightness of the field. If the magnification factor of the cylindrical lens is  $n$  (usually about 150), the brightness is reduced by a factor  $n$ . A spherical lens of the same magnification would reduce the brightness by a factor  $n^2$ . Since lack of brightness is one of the disadvantages of the instrument, the saving of the cylindrical lens is well worth having.

**(d) Compensator method.** It is tedious to count the large number of fringes that pass the crosswire as the gas is admitted into the tube  $T_1$ . The increase in the optical path length of beam 1 is therefore compensated by increasing the optical path length of beam 2 by a measured amount. The two fringe systems are used as an indicator to determine when the two optical paths are equal.

Several methods of compensation have been devised. We shall confine ourselves to one. A fairly thin glass plate is placed in each of the two upper beams; one is fixed and the other can be rotated. When the two plates are parallel the path difference is zero. The path difference is varied by rotating the rod  $R$  (Fig. 7.3) to which the moving plate is attached. This is done by means of a micrometer screw  $M$  which bears against the radial arm  $A$ . In this way small changes in path difference can be made precisely and reproducibly.

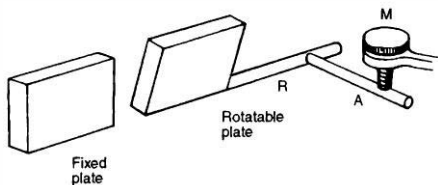


Fig. 7.3. Compensating device.

The path difference produced by the compensating device can be calculated as a function of the setting of the micrometer screw, provided we know the dimensions and refractive index of the rotating plate. But it is a good general principle to prefer an *empirical* to a calculated calibration. So the compensator is calibrated by observations of the fringe system with monochromatic light. The

micrometer screw is rotated and its readings for displacement of 1, 2, 3, ... fringes are noted. The calibration must be done at various wavelengths of incident light.

(e) *White-light fringes.* All the discussion so far has been based on the use of monochromatic light. Clearly the ultimate measurements must be done in this way, because the wavelength is a crucial quantity in (7.1). However, it would be highly inconvenient if we used only monochromatic light, because all the fringes look alike. It would be necessary to sit watching the fringes as the gas was admitted into tube  $T_1$  and to turn the screw on the compensating device at just the correct rate to ensure that the fringe system remained unchanged at all times. If the compensation was not exactly right and we lost sight of the fringes for an instant, we would not know which was the zero-order fringe.

Fortunately this difficulty can be overcome by the use of a source of white light. Each monochromatic component in the source produces its own fringe system with its own spacing, the blue with the narrowest and the red with the widest (Fig. 7.4). What we actually see is the sum of all these fringe systems. For zero path

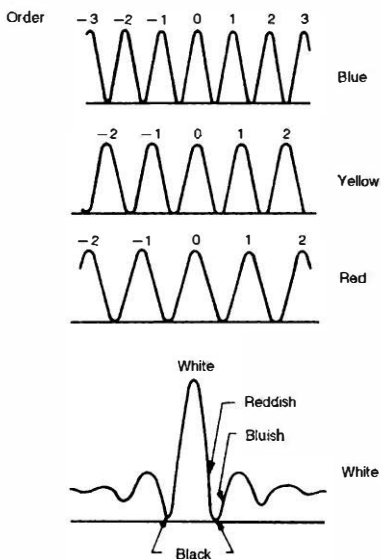


Fig. 7.4. Fringes for blue, yellow, red, and white light. The central white fringe is at zero order.



difference, all the systems give a bright fringe, and the sum is a bright white fringe. But as the path difference increases, the systems get out of step and never get back in step again. (A *finite* set of different wavelengths may produce fringe systems that get back into step, but not an *infinite* set of continuously varying wavelengths.) The total effect is that the fringes on either side of the central white one are coloured. The colouring increases as we go farther out, and the fringes rapidly disappear. So with a white-light source we have a means of detecting zero path difference.

The procedure in the experiment is therefore as follows. We start with both tubes evacuated and a white-light source. The micrometer screw is set approximately so that both sets of fringes coincide. The white light is replaced by a monochromatic source, and the screw is set accurately. The required amount of gas is let into one of the tubes. The screw is then turned to bring the zero-order fringe back into the central position. The fringe is located by means of white light as before, and the final adjustment made with monochromatic light.

(*f*) **Dispersion effect.** A difficulty arises over the white-light method if the dispersion of the gas differs from that of the compensating plate.

Consider the situation when there is no gas in the tube and the compensator is set to zero. The fringe system with the white-light source may be represented by the analytic diagram shown in Fig. 7.5a. This diagram is to be interpreted in the sense that if we draw a horizontal line across it, the crossing points give the fringe spacing at a particular wavelength. The lower we draw the line, the longer is the corresponding wavelength. Thus a horizontal line at the top gives the fringe spacing for blue light and at the bottom for red light.

The total effect of the white light is obtained by collapsing the diagram in the vertical direction. The zero-order fringe, being originally vertical, collapses to a sharp dot, representing the central bright fringe, but the 1st, 2nd, 3rd, ... order fringes collapse to spread out lines which soon overlap, representing fringes that are blurred out.

Now suppose that the gas is admitted and the rotating plate set to compensate for the increase in optical path length. If the dispersions of the gas and the plate are the same, we can achieve compensation at all wavelengths, and the analytic diagram looks identical to Fig. 7.5a. But if the dispersions are different, we cannot achieve simultaneous compensation at all wavelengths. Suppose we set the plate to achieve exact compensation at some wavelength in the middle of the spectrum. The diagram would then look like Fig. 7.5b. The blue, say, is over-compensated and its zero-order fringe has moved to the right; the red is under-compensated and its zero-order fringe has moved to the left. If now we collapse this diagram vertically, it is fringe number 2 that becomes a dot. In other words, the white light fringe is no longer the one of zero order. (In general none of the fringes in Fig. 7.5b is vertical, and the white-light fringe is not of integral order.)

This defect in the ability of the white-light fringes to pick out the zero-order

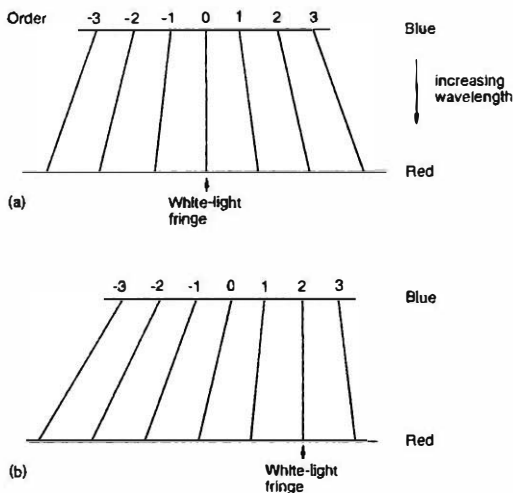


Fig. 7.5. Analytic diagram to show fringe system with white light. A horizontal line gives the fringe spacing at a particular wavelength. Collapsing the diagram vertically gives the result for white light. (a) Dispersions of gas and compensating plate the same, (b) dispersions different.

fringe may be overcome in two ways. The first is to admit the gas gradually so that we can follow the change in the white fringe. The other is to use short tubes to determine a preliminary value of the refractive index, which is sufficiently precise to determine the zero-order fringe when the longer tubes are used – see Ditchburn 1952, p. 279 for further discussion.

(g) **Precision and applications.** We have already said that the smallest path difference that can be detected is about  $\frac{1}{20}\lambda$ . The largest that can be measured is about  $250\lambda$ . We have

$$t \Delta\mu = p\lambda. \quad (7.3)$$

So for a tube 10 mm in length and  $\lambda = 400$  nm, a change in  $\mu$  of  $10^{-6}$  can be detected, and the maximum change that can be measured is  $10^{-2}$ . For a tube 1 m in length, the smallest detectable change is  $10^{-8}$ , and the maximum change is  $10^{-4}$ .

The Rayleigh refractometer is the most precise instrument we have for measuring small changes in refractive index. The refractive index of a transparent mixture – liquid or gas – depends on the proportions of its components, and the refractometer is often the most precise device for determining the proportions or

small changes in them. The instrument finds a variety of applications in this way in physics, chemistry, and biology.

For precise work the temperature must be strictly controlled – to obtain a precision of  $10^{-6}$  for a liquid the temperature must be known to  $10^{-2}$  K. The longer the tube, the more sensitive the apparatus, and the more difficult to obtain reliable results. Therefore we choose the shortest tube that will achieve the required precision in a given situation.

The refractometer was originally designed by Lord Rayleigh (1896) to measure the refractive indices of the recently discovered inert gases, helium and argon. The value of  $\mu - 1$  at s.t.p. is  $3.5 \times 10^{-5}$  for helium and  $28.1 \times 10^{-5}$  for argon.\* As we have seen, the instrument is well able to measure even these low values.

## 7.2 Measurement of resistivity

**(a) Introduction.** For many years the most precise instrument for measuring electrical resistance was the dc potentiometer, but it suffers from the disadvantage that thermoelectric and drift effects give errors that are troublesome to eliminate. For this reason ac methods are now more commonly used for precise electrical measurements. As an example of a current technique we have selected a method developed by Friend and others for measuring the resistivity and the Hall effect in metallic specimens at low temperatures. The design of the apparatus contains a number of interesting features showing what can be done with semiconductor devices. We give here a simplified account of the method. Further details can be found in Friend and Bett 1980. To follow the discussion you will need some basic ideas on devices such as operational amplifiers, binary counters, and so on. These may be found in a number of books on electronics – see for example Horowitz and Hill 1989.

**(b) Description of the method.** The basic experimental arrangement is shown in Fig. 7.6. A sine-wave voltage is generated by digital means at a frequency of  $f_0 = 70$  Hz and fed into a unit which produces an alternating current of the same

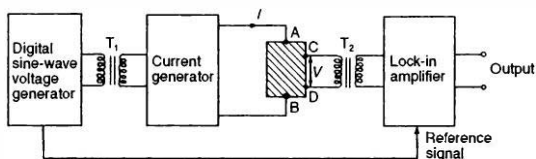


Fig. 7.6. Block diagram of apparatus for measuring the resistivity of a metallic sample (shown shaded).

\* Kaye and Laby 1995, p. 131. The values are for  $\lambda = 589$  nm.

frequency and of constant amplitude. The current  $I$  is passed through the sample via a pair of contacts A and B. The resulting voltage  $V$  developed across another pair of contacts C and D is fed into a unit, known as a *lock-in amplifier*, which acts as a highly tuned detector or demodulator, and gives a dc output proportional to the amplitude of the alternating voltage  $V$ . An essential feature of a lock-in amplifier is the injection of a reference signal with the same frequency  $f_0$  as the signal from the sample. This is provided by the sine-wave voltage generator.

The current  $I$  is determined in a preliminary experiment by passing it through a known resistance and measuring the voltage across the resistance. The lock-in amplifier is calibrated by applying a known alternating voltage to its input terminals and measuring the resulting dc voltage at the output. These measurements are made with a digital voltmeter which can measure dc and ac voltages and also resistance.

To obtain the resistivity of the sample, the measurements are repeated with A and D as the current contacts and B and C as the voltage contacts. An ingenious theorem by van der Pauw (1958) shows that, for a sample in the form of a lamina of uniform thickness, the two sets of measurements, plus the thickness of the lamina, suffice to determine the resistivity of the material – irrespective of the shape of the lamina and the location of the points A, B, C, D.

With this apparatus the experimenters were able to measure the current and voltage with a precision of about 1 part in  $10^4$ . We consider each unit of the apparatus in turn to show how it functions.

**(c) Digital sine-wave voltage generator.** The elements of this unit are shown in Fig. 7.7. A timer circuit produces clock pulses at a frequency of  $256f_0$ . These are fed into an 8-bit binary counter which produces the numbers 0 to 255 in succession. The entire sequence is thus repeated with a frequency  $f_0$ . The numbers are fed into a read-only memory with 256 registers. Each number locates a register, the contents of which – also an 8-bit number – are produced at the output of the memory unit. The contents of register  $n$  are preprogrammed to be

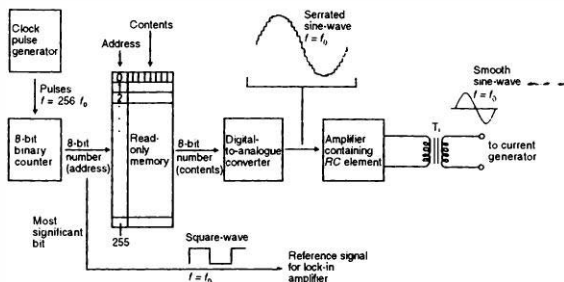


Fig. 7.7. Digital sine-wave voltage generator.

$$c_n = \text{nearest integer to } \left\{ \left[ \sin \left( \frac{2\pi n}{256} \right) + 1 \right] \times 127.5 \right\}. \quad (7.4)$$

The successive integers  $c_n$  (which you can see are in the range 0 to 255) are fed into a digital-to-analogue converter. This is a circuit that produces a voltage proportional to the digital input (Horowitz and Hill, p. 614). The output from the converter is thus a serrated sine-wave as shown schematically in Fig. 7.7. The serrations are removed by an  $RC$  element in the circuit. Finally a 1:1 transformer  $T_1$  removes the dc component, thereby providing a floating voltage signal which acts as the input to the current generator. This signal is a smooth sine-wave of frequency  $f_0$ . Its amplitude may be varied in steps from 0.5 V to 2.5 V.

The reason for using a digital method to produce the sine-wave voltage is that the amplitude of the wave is very stable, being fixed by the digital-to-analogue converter. This unit has a built-in voltage reference – see section (g) – that regulates the output voltage to a few parts in  $10^5$ .

The most significant bit of the 8-bit number from the binary counter is used as a reference signal for the lock-in amplifier. It is a square-wave with frequency  $f_0$ .

**(d) Current generator.** The current generator is shown in the left half of Fig. 7.8. The operational amplifier  $A_1$  has the property that its gain is very high, so the voltage across its input terminals is very small and may be taken to be zero. The voltage across the resistor  $R_1$  is thus equal to the voltage  $V_1$  from the secondary of the transformer  $T_1$ . The current through  $R_1$  is therefore

$$I = \frac{V_1}{R_1}. \quad (7.5)$$

Since the input impedance of the operational amplifier is high and the voltage across the input terminals is small, the current through the input terminals is very small indeed. Therefore virtually all the current  $I$  passes through the sample via the contact points A and B. The resistance  $R_1$  may be varied in steps to give a range of current values from  $10 \mu\text{A}$  to 20 mA.

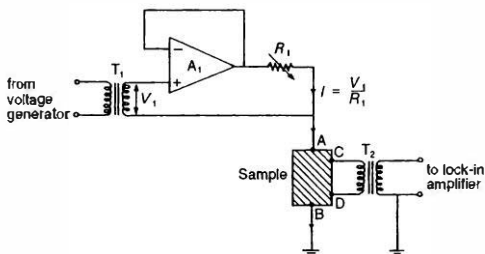


Fig. 7.8. Current generator and sample.

The basic property of the current generator is that the current depends only on the values of  $V_1$  and  $R_1$  according to (7.5). It does not depend on the resistance of the sample or on any contact resistances. So the values of the current, obtained in the calibration measurements when the current is passed through a standard resistor, remain valid when the resistor is replaced by the sample.

The voltage developed across the points C and D is applied to a step-up transformer  $T_2$  with a 1:100 ratio, which serves two purposes. The first is to match the low impedance of the sample to the higher impedance required by the lock-in amplifier. The second is to ensure that the signal applied to the amplifier depends only on the voltage difference  $V_C - V_D$ , and not on the common-mode signal, i.e. the mean voltage  $\frac{1}{2}(V_C + V_D)$ .

(e) **Lock-in amplifier.** A simplified version of this unit is shown in Fig. 7.9.  $M_1$  and  $M_2$  are a pair of MOSFETs,\* which in the present context act simply as a pair of switches. Each switch is closed when the gate of the MOSFET is at a positive voltage and open when the gate voltage is negative. The square-wave reference signal from the voltage generator is applied directly to the gate of  $M_1$  and, via an inverter, to the gate of  $M_2$ . Thus when the reference signal is positive,  $M_1$  is closed and  $M_2$  is open; when the reference signal is negative the reverse is true.

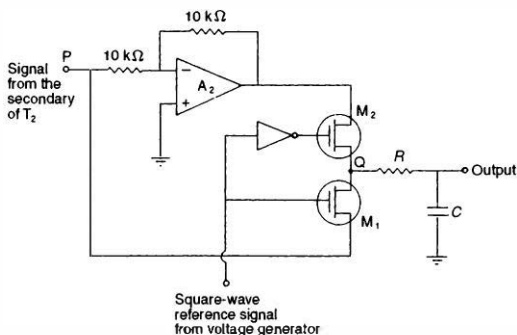


Fig. 7.9. Simplified version of lock-in amplifier.

When  $M_1$  is closed, the signal at P from the secondary of the transformer  $T_2$  appears unchanged at Q. When  $M_2$  is closed, the signal appears at Q with its sign reversed. This is because the non-inverting terminal of the operational amplifier  $A_2$  is earthed. Since the voltage across the input terminals is close to zero, the

\* A MOSFET is a metal-oxide-semiconductor-field-effect transistor. See Horowitz and Hill, p. 117 for a description of the device.

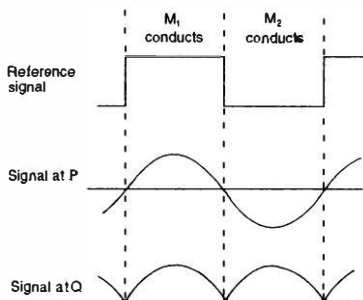


Fig. 7.10. Waveforms in lock-in amplifier. The points P and Q are indicated in Fig. 7.9.

voltage of the inverting terminal is effectively zero – it is a ‘virtual earth’. The potential differences across the two  $10\text{ k}\Omega$  resistors are equal. Therefore, since their common point is at earth potential, the voltages P and Q are equal and opposite. The signal at Q is thus the rectified form of the signal at P – Fig. 7.10. The RC circuit on the right-hand side of Fig. 7.9 acts as a low-pass filter, i.e. the voltage across the capacitor C is, apart from a small ripple, equal to the dc component of the signal at Q, which in turn is proportional to the amplitude of the alternating voltage at P.

**(f) Noise reduction.** In a typical set of measurements the resistance of a small metallic sample varies from about  $10^{-3}\ \Omega$  at room temperature to about  $10^{-6}\ \Omega$  at low (liquid-helium) temperature. To avoid heating the sample, the maximum current that can be used is about 1 mA. The signal from the sample is therefore of the order of nV. The ac method eliminates the effects of thermoelectric voltages which, in the present experiment where leads are taken from room temperature to liquid-helium temperature, are of the order of  $\mu\text{V}$ . However, with such a small signal, noise is a major problem. Apart from Johnson and  $1/f$  noise mentioned in the last chapter, there are effects due to mechanical vibrations of the equipment in the presence of magnetic fields. In general there is also pickup from the mains at a frequency of 50 Hz, but the experimenters avoided this by working at a frequency of 70 Hz.

The purpose of the lock-in amplifier and the low-pass filter, represented by the RC element, is to reduce the noise. Consider the RC element first – Fig. 7.11a. For a sinusoidal wave of frequency  $f$  the relation between the output  $V_C$  and the input  $V_Q$  is (see exercise 7.1)

$$\left| \frac{V_C}{V_Q} \right| = \frac{1}{(1 + 4\pi^2 f^2 C^2 R^2)^{1/2}} \quad (7.6)$$

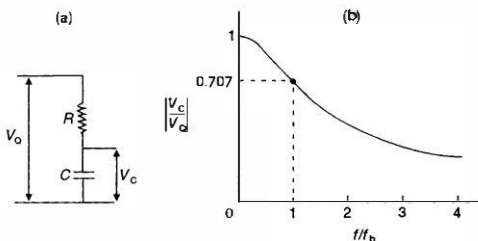


Fig. 7.11. RC low-pass filter.

The relation is plotted in Fig. 7.11b. If we define a frequency  $f_b$  by the relation

$$2\pi f_b CR = 1, \quad (7.7)$$

we can see that

$$\text{for } f < f_b, \quad \left| \frac{V_c}{V_0} \right| > \frac{1}{\sqrt{2}} = 0.707. \quad (7.8)$$

We may take the frequency range  $0 < f < f_b$  as the bandwidth of the filter. The values of  $R$  and  $C$  are chosen so that  $f_b \ll f_0$ . For example, a typical value of  $RC$  is 1 s, giving  $f_b = 0.2$  Hz, which may be compared with  $f_0 = 70$  Hz.

Now consider the lock-in amplifier, which acts as a mixer for the reference and the signal voltages. The reference voltage has a square waveform of fundamental frequency  $f_0$ . Such a waveform contains higher harmonics of frequency  $3f_0$ ,  $5f_0$ , etc, but we may ignore these for the present purpose. Suppose the signal is a sinusoidal wave of frequency  $f$ . Then the result of combining the reference and signal voltages is a wave with frequency components  $f+f_0$  and  $f-f_0$  (see section 6.6). Since the filter only passes frequencies in the range 0 to  $f_b$ , the net result is that the lock-in amplifier plus the filter pass only those signal voltages whose frequencies lie in the range  $f_0 \pm f_b$  or  $(70 \pm 0.2)$  Hz. In other words the combination acts as a highly tuned circuit whose resonance frequency is locked to  $f_0$ .

Let us go back to the experiment. Without the lock-in amplifier the signal would be almost lost in the noise. However, the noise is distributed throughout the frequency spectrum, whereas the signal is entirely at a frequency of 70 Hz. Therefore, with the lock-in amplifier we retain the whole of the signal, but only that part of the noise that lies in the narrow frequency range  $(70 \pm 0.2)$  Hz. The noise is now much less than the signal. This technique for reducing the noise is known as *bandwidth narrowing*.

The technique is a powerful one and finds application in several branches of physics, for example in the measurement of light intensities. Fig. 7.12 shows a schematic arrangement for measuring the transmission of light through a thin



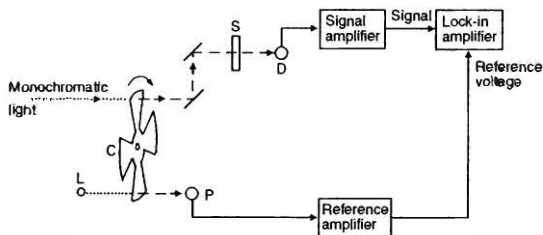


Fig. 7.12. Noise reduction in a measurement of light intensity; ..... steady beam of light; ---- chopped beam.

slab of material as a function of wavelength. A beam of monochromatic light is modulated or chopped by allowing it to impinge normally on a rotating slotted disc C. Fig. 7.12 shows a disc that chops the beam four times per rotation. The frequency of rotation, of the order of 100 Hz, is not critical, provided the modulation frequency  $f_0$  is not near the mains frequency or its harmonics.

The modulated beam passes through the sample S and is detected by a photodetector D, which produces an electrical signal that depends on the intensity of the light falling on it. The signal, a square-wave with frequency  $f_0$ , is amplified and passed into a lock-in amplifier. A light-emitting diode L provides a secondary source of light, which is also chopped by the rotating disc and then detected by a phototransistor P. The output from P, after amplification, is applied as the reference voltage to the lock-in amplifier. You can see that the arrangement acts in the same way as in the resistance measurement. The sample signal and the reference voltage, being modulated by the same rotating disc, always have the same frequency. The effects of extraneous light and of noise in the detecting and amplifying components are much reduced.

**(g) Voltage standards.** We return to the resistivity experiment. The method gives the resistivity in absolute units. The digital voltmeter used to measure the output of the current generator and to calibrate the lock-in amplifier is a commercial instrument capable of giving readings with a precision of about 1 part in  $10^5$ . Such instruments have an internal voltage reference, commonly based on a zener diode. This is a semiconductor diode with the property that when the reverse bias voltage exceeds a certain value (known as the zener voltage) reverse current flows and rises rapidly with increasing voltage. Thus, by arranging that the current through the zener is kept constant within certain limits, the voltage across it is fixed within much closer limits. For diodes used as voltage references, the zener voltage  $V_z$  is about 6 V.

For a given zener diode,  $V_z$  varies with temperature  $T$ . Some values for a particular diode are given in exercise 4.4, p. 42. They correspond to a temperature

coefficient  $dV_z/dT$  of a few  $\text{mV K}^{-1}$ , which is too large for a precise reference device. However, other types of zener diode have lower temperature coefficients – about  $10 \mu\text{V K}^{-1}$  – and are suitable for the purpose.

The reference voltage of a particular instrument may be determined absolutely by comparison with a voltage standard which is ultimately related to those in a laboratory such as the National Physical Laboratory in the United Kingdom or the National Institute of Standards and Technology in the United States. At one time these standards were based on temperature-controlled Weston standard cells, but the present voltage standard is derived from the ac Josephson effect. In the presence of microwave radiation of frequency  $f$ , the voltage across a Josephson junction has the value

$$V = n \frac{h}{2e} f, \quad (7.9)$$

where  $n$  is an integer,  $h$  is the Planck constant, and  $e$  is the elementary charge. Thus the measurement of a frequency, which can be made very precisely, provides an absolute voltage standard. For further details of voltage references see Horowitz and Hill, p. 331. For a discussion of the Josephson effect and its use in voltage standards see Petley 1985.

### 7.3 Absolute measurement of the acceleration due to the Earth's gravity

**(a) Introduction.** In this section we consider a complete experiment, namely, a precision measurement of  $g$ , the acceleration due to the Earth's gravity. Although for many purposes we only require the *variation* of  $g$  from place to place (see p. 113), there are some applications for which we need its *absolute* value. These arise primarily from the need to establish the unit of force in terms of the fundamental units of mass, length, and time. We can measure the mass of a suspended object in terms of the kilogram unit (p. 191). If we know the absolute value of  $g$  at the position of the object, we know the absolute value of the force that the object exerts on its suspension. Apart from practical applications, such as the absolute measurement of pressure and so on, we need the unit of force to establish the unit of electric current, since this is defined in terms of the force between current-carrying conductors. Absolute values of  $g$  are also required for astronomical purposes, such as the calculation of the motion of bodies in the solar system, including nowadays artificial satellites.

Until the Second World War the most precise method of measuring  $g$  was by the use of a reversible pendulum – basically the same as that used by Kater over a century before. However, the effects of small irregularities in the pendulum and its supporting knife-edge placed an ultimate limit of about 1 in  $10^6$  on the precision that could be obtained.\*

The present method is to time the motion of a body in free fall and takes

\* See Cook 1967 for a good introductory article on the absolute determination of  $g$ .

advantage of the very high precision with which small distances and short times can now be measured. The instruments based on this method are of two types – those in which a body is thrown up and measurements are made on the up and down parts of the motion, and those in which the body is simply allowed to fall. Each type has its advantages and disadvantages. The up-and-down instruments are less sensitive to air resistance effects, but care is necessary to eliminate vibrational effects from the body-launching system. The instrument to be described, by Zumberge, Rinker, and Faller 1982, is of the second type.

**(b) Description of the method.** The apparatus is shown schematically in Fig. 7.13. The dropped object  $C_1$  is a cube-corner prism, known as a *retroreflector*, because it has the property that a ray of light, after internal reflection at each of three mutually perpendicular faces, returns travelling in the opposite direction. The principle is illustrated in Fig. 7.14 for the simple two-dimensional case.

$C_1$ , and a similar cube-corner  $C_2$ , act as the ends of an arm of a Michelson

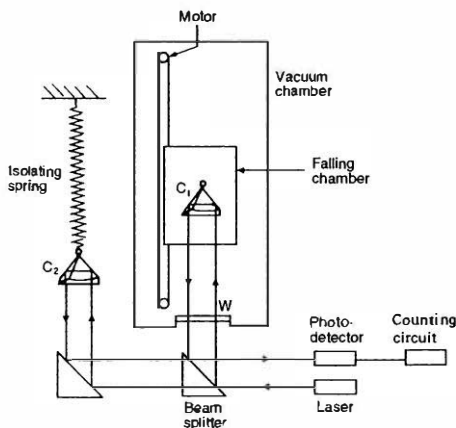


Fig. 7.13. Schematic arrangement of the apparatus of Zumberge, Rinker, and Faller for the absolute measurement of  $g$ .

interferometer. The light source for the interferometer is a stabilized He-Ne laser of wavelength  $\lambda$ . The incident beam is split by a prism, and, after reflection at  $C_1$  and  $C_2$ , the two beams are combined by the prism to give an interference pattern. A small part of the fringe pattern falls on a photodetector which gives an electric signal that depends on the intensity of the light.

During the measurements,  $C_2$  remains at rest, while  $C_1$  falls freely under

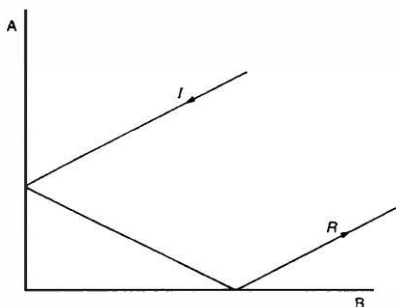


Fig. 7.14. Two-dimensional retroreflector. A and B are two mirrors mounted at right angles. An incident ray  $I$ , travelling in the plane perpendicular to the common axis of the mirrors emerges after two reflections as the ray  $R$  travelling in the direction opposite to  $I$ .

gravity. This causes the fringes to move, and the output from the photodetector oscillates with a period that corresponds to a change in height of  $\lambda/2$  for  $C_1$ .

The total distance that  $C_1$  drops during the measurements is about 170 mm. The value of  $\lambda$  for the He-Ne laser is about 633 nm. The total number of fringes that pass (i.e. the number of sinusoidal oscillations of the output) is therefore

$$N = \frac{170 \times 10^{-3}}{\frac{1}{2} \times 633 \times 10^{-9}} \approx 540000. \quad (7.10)$$

The time of occurrence of every 12 000th fringe is measured by means of a digital clock. Thus about 45 points are obtained in the relation between height  $h$  and time  $t$  as the body  $C_1$  drops. These are fitted to the quadratic relation

$$h = ut + \frac{1}{2}gt^2 \quad (7.11)$$

by the method of least squares. This calculates the values of  $u$  and  $g$  which makes the sum of the squares of the deviations of the experimental values from the theoretical relation in (7.11) a minimum.

The metre is now defined in terms of the second and the speed of light (see p. 191). The wavelength of the laser is thus known in terms of the metre from a measurement of its frequency. The digital clock is calibrated in terms of the caesium standard which is used to define the second – see section 7.4. Hence  $g$  is determined in terms of the metre and the second.

Let us consider what precision might be possible in the experiment. The photodetector can detect a change of about  $2 \times 10^{-3}$  of a fringe movement. So the fractional error in the length measurement is

$$\frac{\Delta h}{h} = \frac{\Delta N}{N} = \frac{2 \times 10^{-3}}{0.54 \times 10^6} \approx 4 \times 10^{-9}. \quad (7.12)$$

The total time of fall is about 0.2 s, which can be measured to about 0.1 ns. Thus the fractional error in  $t$  is

$$\frac{\Delta t}{t} = \frac{0.1 \times 10^{-9}}{0.2} \approx 5 \times 10^{-10}. \quad (7.13)$$

The error in  $g$  is therefore governed by the error in the length measurement, and is potentially about 4 parts in  $10^9$ . Since  $g$  is about  $10 \text{ m s}^{-2}$ , this corresponds to  $\Delta g = 40 \text{ nm s}^{-2}$ .<sup>\*</sup> However, to achieve this accuracy, or something close to it, the systematic errors must be reduced to nearly the same level, and this objective forms the basis of the design of the experiment. We shall now describe the features of the apparatus that reduce the systematic errors to the required levels.

(c) *Falling chamber.* It is clearly essential to eliminate as far as possible all non-gravitational forces on the dropped object. The two most important here are air resistance and electrostatic forces. Air resistance is reduced by reducing the pressure in the chamber in which the object falls. However, at very low pressures, electric charge, which tends to build up on the object, cannot leak away, thus giving rise to electrostatic forces. The experimenters overcame this difficulty by allowing the object to fall inside a chamber which was itself falling.

The mechanism is shown in Fig. 7.15. Light from a light-emitting diode L is focussed by a sphere S attached to the cube-corner  $C_1$  and falls on a detector. This is a device with two elements  $D_1$  and  $D_2$ , each of which produces an electric current which depends on the intensity of the light falling on it. Both the light-emitting diode and the detector are fixed to the chamber. If the dropped object  $C_1$  falls slightly faster than the chamber, light from L falls on the element  $D_1$ ; if  $C_1$  falls slightly slower than the chamber, the light falls on the element  $D_2$ . The difference in the current outputs from  $D_1$  and  $D_2$  is amplified and made to drive a motor which controls the rate of descent of the chamber. Output from  $D_1$  increases the acceleration of the chamber; output from  $D_2$  decreases it. Thus the whole device is a servo system (see section 6.8) for ensuring that the chamber falls with the dropped object  $C_1$ . Notice that  $C_1$  is falling freely and is unaffected by the action of the servo system, which acts only on the chamber. The chamber is vented to the vacuum chamber surrounding it, so that the air pressure in the two chambers is the same.

The falling chamber device has three main advantages. Firstly, it avoids the need for very low pressure to reduce air resistance. The dropped object and the air molecules in the chamber are all falling freely together. The absence of *relative* motion means that there is almost no force from the air molecules. In fact the

\* We continue to use SI units. However, a unit of acceleration commonly used in gravitational work is the gal.  $1 \text{ Gal} = 10^{-2} \text{ m s}^{-2}$ , so  $1 \mu\text{Gal} = 10 \text{ nm s}^{-2}$ .

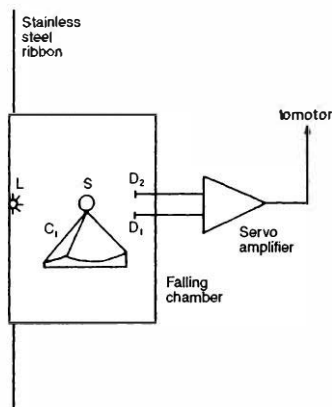


Fig. 7.15. Servo control of falling chamber. The motor, controlled by the output from the servo amplifier, drives the stainless steel ribbon to which the falling chamber is attached.

experimenters still had to work at a reduced pressure (about  $10^{-5}$  mm Hg). This was not due to air resistance, but to the effects of pressure and temperature gradients across the dropped object, which though small are nevertheless significant at the very high precision of the experiment.

The second advantage of the falling chamber is that, being made of conducting material, it shields the dropped object from external electrostatic fields. Lastly, the chamber can be controlled so that it gently arrests the fall of the dropped object at the end of the measurements and returns it quickly to its starting point. This means that many sets of measurements can be made in rapid succession.

**(d) Long-period isolator.** So far we have considered only the motion of the dropped object  $C_1$ , but we need to consider also the cube-corner  $C_2$  which acts as a reference and which is assumed to be at rest. How is this ensured?  $C_2$  must be supported in some way, and its support must ultimately be related to a point on the surface of the Earth. But this point will in general suffer acceleration due to both man-made and seismic vibrations. It is a reasonable assumption that these vibrations have no coherent phase relation to the times of successive drops of  $C_1$ . In other words, they give random and not systematic errors. However, there is no point in striving to reduce the systematic errors in an experiment if there remain substantial random errors. Suppose, for example, that for a single drop of  $C_1$  the error in  $g$  due to the vibrations in the support of  $C_2$  is 100 parts in  $10^9$ , and that it is desired to reduce this error to 3 parts in  $10^9$ . Since the error in the mean is

$1/\sqrt{n}$  times the error in a single reading, the number of drops required to reduce the error to the desired value is about 1000. It is obviously preferable to reduce the error in a single reading, rather than to rely on the inefficient effect of the  $1/\sqrt{n}$  factor, and the experimenters achieved this by means of a servo-controlled spring.

Suppose that the cube-corner  $C_2$  is suspended from a spring to form an oscillating system of natural frequency  $f_0$ . If a sinusoidal motion of constant amplitude and variable frequency  $f$  is applied to the upper end of the spring, it is readily shown that, when the damping is small, the amplitude of the displacement of  $C_2$  for the forced oscillations is approximately proportional to  $1/(f_0^2 - f^2)$ . (When  $f = f_0$  this expression becomes infinite, but in practice the damping force keeps the amplitude finite.) The important point in the present application is that, for  $f \gg f_0$ , the amplitude of the motion of  $C_2$  becomes small, i.e.  $C_2$  is effectively isolated from the motion of the upper end of the spring. For seismic vibrations the period is typically about 6 s. To isolate  $C_2$  it is therefore necessary that the period of free oscillation of the spring should be about 60 s. This would require a spring with a length of about 1 km, which is clearly impracticable.

However, the experimenters were able to obtain an effective period of 60 s from a much shorter spring by electronic means. The principle of the method is as follows. Suppose we have a long spring of length  $L$ , fixed at its upper end, with a body attached to the lower end  $Q$  – Fig. 7.16. As the body makes vertical oscillations with amplitude  $X$ , a point on the spring oscillates with amplitude proportional to its distance from the point of suspension. Thus a point  $P$ , distance  $l$  from  $Q$ , has amplitude  $(1 - l/L)X$ . Now suppose we have a second spring of length  $l$  of the same material and with the same body suspended at its lower end. If, as the body oscillates with amplitude  $X$ , the upper end is made to oscillate with amplitude  $(1 - l/L)X$ , the whole motion of the spring is the same as the lower part  $PQ$  of the first spring, and in particular the frequency of the motion is the same as that of the first spring.

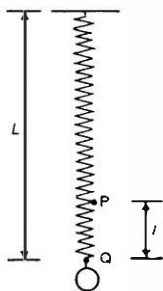


Fig. 7.16. Vertical oscillations of a body on a spring.

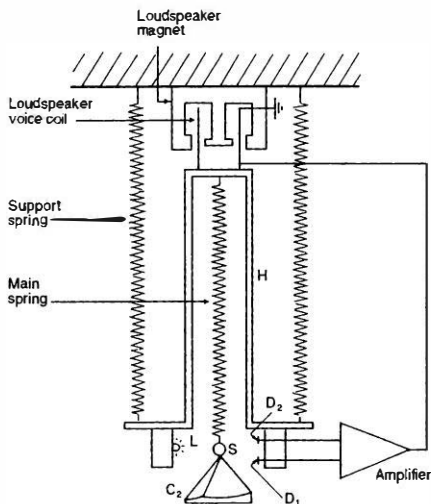


Fig. 7.17. Long-period isolation device.

The upper end of the spring is given the required motion by a servo system very similar to the one controlling the falling chamber. The upper end of the spring is attached to a cylindrical housing  $H$  – Fig. 7.17. Fixed to the bottom of the housing is a light-emitting diode  $L$  and a photodetector with two sensing elements  $D_1$  and  $D_2$ . As before, the sphere  $S$  attached to the cube-corner  $C_2$  acts as a lens, and a signal arrives at the amplifier that depends on the displacement of  $C_2$  relative to the housing  $H$ . The output from the amplifier is fed into the voice coil of a loudspeaker, to which the upper end of the housing is attached. The gain of the amplifier is adjusted to make the amplitude of the oscillation of the housing some required fraction of the amplitude of the body  $C_2$ . The closer the fraction is to unity, the larger is the period of oscillation of the spring. In this way the experimenters were able to use a spring of length 1 m whose period of oscillation corresponded to a spring of length 1 km.

(e) *Other errors.* The experimenters took many other precautions to reduce systematic errors, of which we shall mention only two. The first concerns the optical path. Clearly the light beams must be vertical; otherwise the length measurements will not give the height  $h$ , but  $h \cos \phi$ , where  $\phi$  is the angle between the light beams and the vertical. There is a subtle effect which needs to be considered in this connection. The window in the vacuum chamber ( $W$  in Fig.



7.13) is tilted slightly in order to avoid reflection of the light back to the laser. This by itself does not cause a deviation of the beam from the vertical, because the two surfaces of the window are parallel. However, the difference in air pressure on the two sides means that the refractive indices of the air on the two sides are slightly different, and this, together with the tilt, does give a deviation. Fortunately, the effect is negligible, provided the angle of incidence of the light is close to zero.

A second type of error arises from the electronic circuits that amplify the oscillating electric signal from the detector registering the passage of the interference fringes. As the dropped object descends, the frequency of the signal increases linearly with time, from about 1 MHz at the beginning of the measurements to about 6 MHz at the end. The amplifying circuits produce phase shifts which depend on frequency. It may be shown that phase shifts that vary linearly with frequency do not affect the results – only a non-linear variation gives rise to error. The phase changes depend on the bandwidth of the amplifier.\* The experimenters found that a bandwidth from zero to about 30 MHz was necessary to reduce the error to an acceptable level.

**(f) Results.** The apparatus was tested in several places, including Boulder, Colorado, and Sèvres, near Paris. An example of a series of measurements made at Boulder in May 1981 is shown in Fig. 7.18. The results show the variation of  $g$  with time, at the same place, due to the tidal redistribution of water in the oceans. This variation has several components with different periods, the two most prominent having periods of a day, and half a day.<sup>†</sup> They are clearly seen in the figure. Each

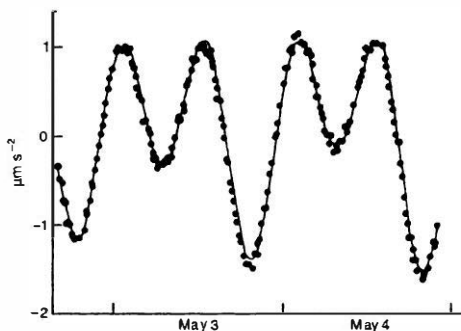


Fig. 7.18. The tidal variation of  $g$  at Boulder, Colorado measured with the apparatus of Zumbege, Rinker, and Faller in May 1981. The curve shows the theoretical variation.

\* See Horowitz and Hill 1989, p. 243.

<sup>†</sup> See Cook 1973, chapter 4, for the theory of the variation of  $g$  due to the tides.

experimental point is the result of a set of 150 drops. The rms deviation of the points after the variation of the tides has been removed is  $60 \text{ nm s}^{-2}$ .

Measurements taken at different places indicated an average error of about  $100 \text{ nm s}^{-2}$ , which corresponds to a fractional error of  $\Delta g/g = 10^{-8}$ . It may be noted that this error is about 100 times smaller than that of the most accurate reversible pendulum. A striking illustration of the very high precision of the apparatus is that its error is equivalent to the change in  $g$  due to a change in height of about 30 mm (!) at the surface of the Earth – see exercise 7.3.

Since this experiment in 1982 the instrument has been improved (Niebauer *et al.* 1995), and a number of *absolute gravimeters*, as they are called, have been constructed and operated in various parts of the world. In 1994 a comparison was made of the results of eleven instruments at five sites, which showed that the fractional error for the measurement of  $g$  had been reduced to about  $3$  to  $4 \times 10^{-9}$  (Marson *et al.* 1995).

An interesting experiment was reported by Peters, Chung, and Chu in 1999. They measured the gravitational acceleration of falling *atoms* with a fractional error of  $3 \times 10^{-9}$ , and showed that the value obtained agreed with the value for a falling macroscopic object to within 7 parts in  $10^9$ . The experiment involved very sophisticated atomic techniques; you are referred to the paper in *Nature* for the details.

## 7.4 Measurement of frequency and time

**(a) Introduction.** Frequency – and its reciprocal, time – can be maintained and measured to a higher degree of precision than any other quantity in physics. This precision is not only the basis of some fundamental definitions, but also has a number of important applications.

Several methods are available for producing oscillations of a standard frequency. The simplest is to make use of a quartz crystal, which, by virtue of the piezoelectric effect, can act as a highly tuned circuit and produce an electromagnetic oscillation whose frequency is a characteristic of the geometry of the crystal and its elastic properties. Crystals giving a frequency stability of 1 part in  $10^6$  are readily available. If the temperature of the crystal is controlled, a stability of about 1 part in  $10^{10}$  may be obtained.

**(b) The caesium atomic clock.** Still higher stability is obtained from an oscillator whose frequency is controlled by transitions between two energy levels in an atom. The atom that has proved most suitable is caesium. A full explanation of the caesium clock needs quantum mechanics,\* but we outline the basic ideas here. The caesium atom has only one stable isotope,  $^{133}\text{Cs}$ . The spin of its nucleus is  $\frac{7}{2}$ . It has a single valence electron, which has spin  $\frac{1}{2}$ , and zero orbital angular

\* The basic physics can be found in a number of textbooks – see for example Gasiorowicz 1996, chapters 15 and 22, and Bransden and Joachain 1983, p. 232.

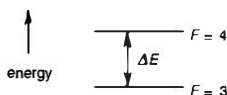


Fig. 7.19. The two lowest energy states of the  $^{133}\text{Cs}$  atom.

momentum in the ground state. The quantum number for the resultant of the nuclear and electronic spin angular momenta is denoted by  $F$ , and, from the rules for adding angular momentum in quantum mechanics, takes values  $\frac{7}{2} + \frac{1}{2} = 4$ , and  $\frac{7}{2} - \frac{1}{2} = 3$ . The spin motions of the nucleus and the electron give rise to magnetic dipole moments. The  $F=3$  state corresponds to the two magnetic dipoles being in the same direction, and  $F=4$  to the two dipoles being in opposite directions. The two states have slightly different energies, the  $F=3$  state being lower – Fig. 7.19. Suppose we have a system of caesium atoms, some in one state and some in the other, and shine electromagnetic radiation of frequency  $f_0$  on it. If  $f_0$  satisfies the relation

$$\Delta E = hf_0, \quad (7.14)$$

where  $\Delta E$  is the energy difference between the two states, and  $h$  is the Planck constant, transitions will occur. Some atoms in the  $F=4$  state change to the  $F=3$  state, and vice versa. If the frequency of the radiation does not equal  $f_0$ , transitions do not occur, and the atoms remain in their initial state.

The elements of the caesium clock are shown schematically in Fig. 7.20. The caesium is in an oven at a temperature of about  $100^\circ\text{C}$ , which vaporizes the metal. The emerging beam, which contains atoms in both the  $F=4$  and  $F=3$  states in nearly equal numbers, passes into a magnet – A in the diagram – which is constructed to have a very inhomogeneous field. (It is often referred to as a *Stern–Gerlach* magnet after two German physicists who pioneered its use.) The

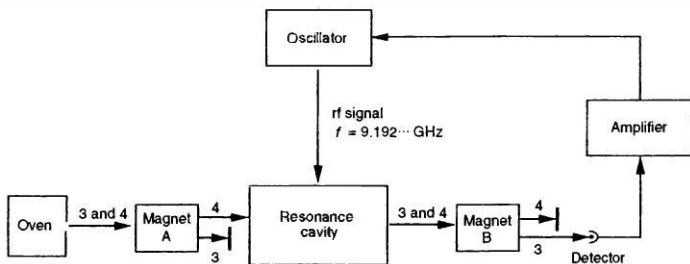


Fig. 7.20. Schematic representation of the caesium clock. The values of  $F$  are shown next to the beam paths.

inhomogeneous field produces a deflecting force on an atom that depends on its magnetic dipole moment. The atoms in the two quantum states have different values of magnetic dipole moment, and the magnetic field deflects them in opposite directions. The two sets of atoms emerge from the magnet along different paths, and the  $F=3$  atoms are then blocked. The  $F=4$  atoms pass into a cavity where they are subjected to a radio-frequency oscillation of frequency  $f_0$ . This causes transitions so that some of the atoms leaving the cavity are in the  $F=3$  state. The beam now passes through a second magnet B, identical to magnet A. The atoms in the  $F=4$  and  $F=3$  states again emerge along different paths, and this time the  $F=4$  states are blocked. The atoms in the  $F=3$  state, i.e. those that have made a transition in the cavity, strike a tungsten detector, which gives a signal that depends on the rate of arrival of the atoms.

If the frequency of the oscillator producing the radiation in the cavity drifts from  $f_0$ , the rate of transitions in the cavity drops and so does the output from the detector. An electronic arrangement determines whether the frequency is less than or greater than  $f_0$ , and sends a signal to the oscillator which brings the frequency back to the correct value. We thus have a servo system that keeps the oscillator frequency tightly locked to the value  $f_0$ .

The precision of present caesium clocks is better than one part in  $10^{14}$ . The longer the time the caesium atoms are in the resonant cavity, the more precise is the frequency. In recent experiments this time has been increased by slowing the atoms down by radiative forces from photons in a laser beam, a process known as *laser cooling* -- see Cohen-Tannoudji 1998. They are then thrown upwards and allowed to fall back under gravity along the same path. This 'fountain' technique has so far yielded a further factor of 10 in precision. Experiments are in progress to use the transitions between states of *ions* trapped by a set of electromagnetic fields to stabilize the frequencies of oscillators. These promise even higher precision, but that is for the future.

**(c) Definition of the second and other applications.** The caesium clock is so stable that since 1967 it has been used to define the second. Prior to then the second was defined in terms of the Earth's yearly motion round the Sun. However, neither this nor the Earth's daily rotation is as stable as a caesium-controlled oscillator. The definition of the second is made by assigning the specific value of 9.192631 770 GHz to  $f_0$ . The value was chosen to make the second agree with its previous definition.

Because frequency and time can be measured so precisely, other definitions are tied to them. Thus the metre is defined as the distance travelled by light in a certain time. We try to measure a physical quantity by relating it to frequency. For example, the Josephson effect can be used to relate voltage and frequency, thereby providing a very precise and reproducible volt in the SI system of units -- see p. 86.

Another example is the method of *proton magnetic resonance* for measuring a

magnetic field. The proton has a magnetic dipole moment  $\mu_p$ , and in a magnetic field  $B$  the dipole may be parallel or antiparallel to the field, the two states differing in energy by an amount  $2\mu_p B$ . If an alternating electromagnetic field of frequency  $f$ , where

$$hf = 2\mu_p B, \quad (7.15)$$

is applied to a system of protons, for example a small sample of water, transitions occur between the two states. The system absorbs energy from the alternating field, and the resonance is readily detected. The quantity  $\gamma_p = 2\mu_p/\hbar$  is known as the *gyromagnetic ratio* of the proton. ( $\hbar = h/2\pi$ .) Thus (7.15) may be written as

$$\omega = 2\pi f = \gamma_p B. \quad (7.16)$$

The value of  $\gamma_p$  is known to about 4 parts in  $10^8$  – see p. 192 – so a measurement of the frequency gives the value of the magnetic field to a high degree of precision.

**(d) Time scales.** In addition to providing a definition of the second, the very high stability of atomic clocks allows us to define a standard, universal time scale. The outputs of over 200 atomic clocks from laboratories all over the world are sent to an organization known as the *International Bureau of Weights and Measures* (BIPM) outside Paris, which defines a time scale from the weighted mean of all the clocks. This is known as *International Atomic Time* (TAI). We could have used this time scale, which is precise to about one part in  $10^{14}$ , as a universal time scale, but it is convenient for many purposes to have a scale related to the daily rotation of the Earth, even though its period varies and is slowly increasing. To meet the requirement of a very precise time scale, which is also related to the mean rotation of the Earth, a new time scale known as *Coordinated Universal Time* (UTC) has been devised. This is synchronous with TAI, except that on certain specific occasions a *leap second* is added to UTC.\* The occasions are determined from observations made at an observatory in Paris which accurately monitors the rotation of the Earth. Twenty-two such leap seconds have been added from 1972 to 1999. (In principle a leap second may also be subtracted, but this has not so far happened.) The net result is that UTC is determined with the precision of atomic clocks, but is always within one second of a time scale fixed by the rotation of the Earth. The announcement of the insertion of a leap second is published in advance, so the whole world has an agreed official time scale. UTC is used in scientific work, such as astronomy, where groups in different laboratories around the world need to coordinate their observations.

A number of radio stations throughout the world broadcast signals with the frequency of the carrier maintained by an atomic standard. In the United Kingdom, for example, the National Physical Laboratory provides a signal known as MSF which is broadcast from Rugby at a carrier wave frequency of

\* On such an occasion someone holding a watch that ticked every second according to TAI would observe 61 seconds in that particular minute of UTC.

60 kHz, stabilized to 2 parts in  $10^{12}$ .\* The signal may be fed into a laboratory oscillator (or even nowadays into an inexpensive domestic radio clock) which is then stabilized to the same degree.

## 7.5 The Global Positioning System

Not only have caesium clocks become more accurate over the last three decades, they have also become more stable in different environments, and their size has been reduced. This has led to a very important application of atomic timekeeping, known as the *Global Positioning System* (GPS), which enables objects to be located on the surface of the Earth with very high precision.

**(a) Principle of the system.** The system is based on twenty-four satellites in orbit at heights of about 20000 km. Each satellite contains an atomic clock and radiates radio-frequency signals whose modulating patterns are locked to a common time scale. The satellites are spaced so that, at any point on the ground, a receiver can at all times pick up the signals from at least four of them. The receiver contains a clock and a computer, which calculates the time taken for each signal to travel from the satellite to the receiver. Since the signals are travelling with the speed of light, this gives the distances from each satellite to the receiver. The satellites also broadcast their positions. Putting all this information together the computer can determine the location of the receiver on the surface of the Earth to within about a metre!

We illustrate the principle in two dimensions in Fig. 7.21. If the time taken by a signal from a satellite at  $S_1$  to reach the receiver is  $t_1$ , and the time for a signal from a satellite at  $S_2$  is  $t_2$ , then the receiver is at a distance  $r_1 = ct_1$  from  $S_1$  and a

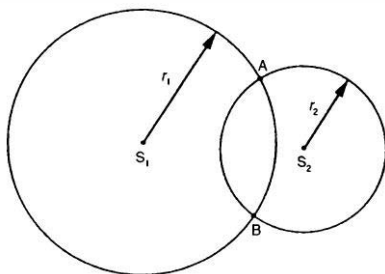


Fig. 7.21. The principle of the Global Positioning System. The receiver is at measured distances  $r_1$  and  $r_2$  from the satellites  $S_1$  and  $S_2$ , and is therefore at either A or B.

\* The short-term stability of the received signal may not be as good as that of the transmitted signal, owing to variations in ionospheric conditions.

distance  $r_2 = ct_2$  from  $S_2$ , where  $c$  is the speed of light. Thus the receiver lies on the intersection of the two circles centred on  $S_1$  and  $S_2$  with radii  $r_1$  and  $r_2$ . There are two intersections, A and B in the figure, but a very approximate value of the position enables one to be ruled out. In three dimensions we need signals from three satellites to locate the position of the receiver. The receiver lies at the intersection of three spheres centred on the positions of the three satellites with radii determined by the travel times of the three signals. The three spheres intersect at two points. In general, only one of them is on the surface of the Earth; the other is either deep inside the Earth or in far outer space, and so can be ruled out.

**(b) Measurement of the travel time of the signal.** The travel times of the signals are determined by an ingenious method. The signals from each satellite have a pseudo-random pattern, i.e. they appear to have a random sequence, but this sequence is in fact determined by a formula. Each sequence, together with its time of transmission, is built into the memory of the computer in the receiver. The situation is shown schematically in Fig. 7.22. (a) is part of a transmission sequence from a satellite. When the transmitted pattern reaches the receiver it is delayed by the travel time  $t_s$  and appears as (b). The computer determines the value of  $t_s$  by finding out how much (b) must be advanced to bring it into maximum correlation with, i.e. to match, its copy of (a).

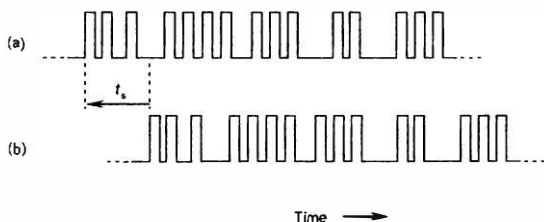


Fig. 7.22. Diagram showing how the travel time  $t_s$  of the signal is measured. (a) is the signal transmitted by the satellite. An identical copy is stored in the receiver's computer. (b) is the signal when it reaches the receiver. The computer determines the value of  $t_s$  by finding how much (b) must be advanced to bring it into maximum correlation with the copy of (a).

**(c) Correction to the receiver clock.** Signals from three satellites are sufficient to determine a position in three dimensions. However, if the information came from only three satellites, the clock in the receiver would have to be as accurate as those in the satellites. But putting an atomic clock in each receiver would be prohibitively expensive. With signals from four satellites it is possible to use a relatively

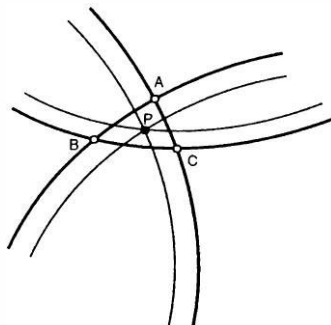


Fig. 7.23. Diagram illustrating the correction of the error in the receiver's clock for the case when the clock is running fast. The original circles (thick lines) intersect at three points. The computer calculates the clock correction necessary to reduce the radii of the circles so that they intersect at a single point.

cheap quartz clock of lower accuracy, and to use the information of the signals to correct the clock.

The method is demonstrated in two dimensions in Fig. 7.23. For two dimensions we need signals from three satellites to make the clock correction. The figure shows the situation if the receiver clock is running fast. As a result the calculated values of the travel times, and hence the radii of the circles, are too large. The circles (thick lines in the figure) intersect at the points A, B, and C instead of at a single point. It is a relatively simple calculation for the computer to find the correction to the clock that reduces the three radii so that the circles intersect at the single point P, which is the correct location of the receiver.

**(d) Features of the system and applications.** The Global Positioning System was constructed by the United States Department of Defence for military purposes. The first satellite was put into orbit in 1978. Subsequent satellites had improved facilities, and each one now has four atomic clocks on board. Light travels  $3 \text{ m}$  in  $10 \text{ ns}$ . So to achieve a precision of  $1 \text{ m}$  in location, the times have to be known to a few nanoseconds. Caesium clocks are stable to within a few parts in  $10^{14}$  per day, which amounts to a few nanoseconds per day. The clocks in the satellites are monitored by a set of control stations located in different parts of the world at known positions on the ground. Signals are sent back to the satellites, which adjust their clocks to give the required degree of synchronism. The time scale used is based on atomic time (TAI).

The control stations also monitor the positions of the satellites by reflecting laser beams from them and measuring the travel time of the signal. The satellites



are equipped with cube-corner reflectors – see p. 87 – which reverse the direction of the laser beam whatever the orientation of the satellite. The measured ranges are accurate to a few centimetres!

The United States Government has made the Global Positioning System available for civilian, as well as military, purposes. It is used in navigation by ships, aircraft, and ground vehicles, including domestic cars, and also by hikers and boating enthusiasts. But it has many other uses. It is so precise that it can measure small deformations of the Earth's crust, providing information on impending earthquakes. It can also be used in weather forecasting. Water in the atmosphere affects the travel times of the radio signals. So if the location of the receiver is known, the atmospheric water content can be measured. There are many other uses for the system, and their number is growing rapidly. A large number of commercial GPS receivers are available, catering for the different types of use, and there is a magazine *GPS World* devoted to the system.

You will find details of the topics in the last two sections in a number of publications. A very readable book on atomic clocks and time scales is *Splitting the Second* by Jones 2000. A full account of the physics of timing devices from sundials to quartz clocks, caesium clocks, masers, and ion traps will be found in *The Quantum Beat* by Major 1998. For a popular account of the Global Positioning System see Herring 1996.

### Exercises

- 7.1 Prove the result in (7.6).
- 7.2 A zener diode with  $V_z = 6$  V, operated at a current of  $I_z = 1$  mA, has a dynamic resistance  $dV_z/dI_z = 3$   $\Omega$ . If the current varies by 2%, what is the fractional change in the voltage?
- 7.3 Estimate the increase in height at the surface of the Earth corresponding to a reduction in  $g$  of 1 part in  $10^8$ .

## 8 Experimental logic

### 8.1 Introduction

Systematic error is just a euphemism for experimental mistake. Such mistakes are broadly due to three causes:

- (a) inaccurate instruments,
- (b) apparatus that differs from some assumed form,
- (c) incorrect theory, that is, the presence of effects not taken into account.

We have seen the remedy for the first – calibrate. There is no blanket remedy for the other two. The more physics you know, the more experience you have had, the more likely you are to spot the effects and hence be able to eliminate them. However, there are ways of making measurements, of following certain sequences of measurements, which automatically reveal – and sometimes eliminate – certain types of error. Such procedures form the subject of the present chapter. Some are specific, others are more general and add up to an attitude of mind.

Finding and eliminating a systematic error may sound a negative, albeit desirable, object. But there is more to it than that. The systematic error that is revealed may be due to a phenomenon previously unknown. It is then promoted from an ‘error’ to an ‘effect’. In other words, by careful measurement we may make discoveries and increase our understanding of the physical world.

### 8.2 Apparent symmetry in apparatus

It is a good rule that whenever there is an apparent symmetry in the apparatus, so that reversing some quantity or interchanging two components should have no effect (or a predictable effect – see the second example), you should go ahead and make the change. Two examples will illustrate the point.

Consider an experiment, for example, the measurement of the thermal conductivity of a material, in which we need to measure the temperature difference  $\Delta\theta$  between two points P and Q. Suppose we do this by measuring the temperature at P and Q with a pair of similar thermometers. Symmetry says that interchanging the two thermometers should not affect the result. We interchange them and find that it does, thereby discovering that the thermometers are not

reading correctly. If  $\Delta\theta$  is small, its value from one pair of temperature readings could be seriously wrong. Interchanging the thermometers and taking the mean of the two values of  $\Delta\theta$  considerably reduces the error. (If  $\Delta\theta$  is small, a better method still is to measure it directly and thus avoid the unsatisfactory procedure of taking the difference between two nearly equal quantities. This could be done by placing platinum resistance thermometers at P and Q and connecting them in the opposite arms of a Wheatstone bridge.)

The second example is the Wheatstone bridge. Look at the circuit in Fig. 8.1.  $R$  is an unknown resistance, and  $S$  is a standard of known value. The value of  $R$  is obtained by finding the value of AB at balance. Denoting it by  $x_1$  and the value of AC by  $l$ , we have

$$\frac{R}{S} = \frac{x_1}{l - x_1}. \quad (8.1)$$

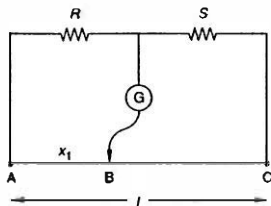


Fig. 8.1. Wheatstone bridge.

Symmetry says that if we interchange  $R$  and  $S$ , the new value of AB should be

$$x_2 = l - x_1. \quad (8.2)$$

We make the change and get a different value. The operation has revealed the presence of end effects.

### 8.3 Sequence of measurements

The *order* in which measurements are made can be very important, as the following example illustrates. Three students are asked to find how the terminal velocity of a sphere falling in a liquid varies with its diameter. They are given a set of four ball bearings of various sizes and a large tank of glycerine.

Student X takes the largest ball and measures its terminal velocity five times, then takes the next largest and does the same, and carries on until he reaches the smallest. He gets a poor result. Why? Because the laboratory has been warming up during the measurements, and so has the glycerine. The viscosity of glycerine, like that of most liquids, drops rapidly with increase in temperature. The terminal

velocity depends on the viscosity of the liquid as well as the diameter of the ball. Therefore, since the average viscosity was different for each ball, the results do not give the variation of terminal velocity with diameter alone.

Student Y knows more physics than X and knows about the variation of terminal velocity with viscosity and hence with temperature. He therefore builds a device for keeping the temperature of the glycerine constant. He makes the measurements in the same sequence as X and gets a much better result, but it is still incorrect. Why? Because unknown to Y the clock that he is using to measure the terminal velocity is gradually slowing down, and this gives an effect systematically different for the four balls.

The third student Z is as ignorant as X about the effect of temperature on the measurements, and his clock is just as poor as Y's, but he gets a very good result. This is due to the sequence in which he – instinctively – makes his measurements.

Denote the four balls by A, B, C, D. Suppose that instead of five successive measurements for ball A, followed by five for ball B and so on, the measurements are made in the order

ABCDABCD...

Now instead of ball A being measured with the high and D with the low viscosity liquid, all four balls are being measured at a high value and then again at a lower value and so on. However, although this sequence reduces the systematic error considerably, it is still true that for each set of four measurements A comes at the high and D at the low viscosity end. So even better is the sequence

ABCDDCBA,

which is repeated as many times as the total time permits. This is the way student Z makes his measurements. (An extra precaution would be to make the next sequence BCDAADCB and so on.) You can see that over the entire sequence of measurements, the effects of a smooth variation with time of the viscosity of the liquid, or of the accuracy of the clock, or indeed of any factor other than the diameter of the ball, will probably be small.

Note that even Z's method can be improved upon. His ignorance of the temperature effect is hardly a merit. The measurements in this experiment are so sensitive to the temperature of the liquid, that a competent experimenter would not only adopt Z's sequence, but would also measure the temperature from time to time to check that there was no chance correlation between the temperature variation and the sequence of the diameters.

## 8.4 Intentional and unintentional changes

In an experiment to measure the effect of varying one quantity, we obviously try to keep all other quantities constant. However, there is always the possibility of variations in the latter, and in the last section we described a method for reducing

the effects of such variations. The method is very effective, but is only applicable when the unwanted variations are not caused by or related to the quantity that we wish to vary. This is clearly the case in the previous example. Neither the temperature of the glycerine nor the accuracy of the clock depend on which ball we decide to drop for the next measurement.

However, consider the following experiment. We wish to investigate the change in the dimensions of a ferromagnetic material caused by the application of a magnetic field, a phenomenon known as *magnetostriction*. An iron rod is placed in a solenoid, and its length is measured as a function of the current through the solenoid, the current being a measure of the magnetic field.

Now the change of length in magnetostriction is small – the fractional change for complete magnetization is about  $5 \times 10^{-5}$  – and so in order to measure it accurately, we must keep the temperature of the sample constant; otherwise thermal expansion will swamp the magnetic effect. When we increase the current through the solenoid we increase the heat generated in it, and this may raise the temperature of the specimen. The method of the last section is quite irrelevant here – the quantity we are varying is *causing* the unwanted variation. What we have to do is to ensure that the current through the solenoid does *not* affect the temperature of the specimen, for example, by winding the solenoid on a water-cooled former.

The converse effect is also a potential source of error. The current through the heating coil of a furnace may give rise to a magnetic field, which in turn may affect the measurements.

## 8.5 Drift

In section 8.3 we had an example of slow systematic variation or *drift* during an experiment. Apart from temperature, other common quantities that are liable to vary are atmospheric pressure and humidity, the voltage of a battery, the mains voltage and even its frequency.

Choosing an appropriate sequence for the measurements is one way of reducing the effects of these variations, but often we wish to prevent or at least minimize the variations in the first place. This is usually done by various negative feedback and servo devices (sections 6.7 and 6.8).

Section 8.3 also provided an example of *instrumental* variation. We should always have it in mind that instruments are liable to drift – their zero errors may change and so may their sensitivities. It may therefore be necessary to calibrate an instrument more than once – indeed many times – during an experiment.

Notice that the calibration operation itself may form part of a sequence which can give rise to a systematic error. Suppose, for example, we are comparing two voltages  $V_1$  and  $V_2$  by means of a potentiometer. The emf of a potentiometer cell tends to fall with time. So if, after standardizing the instrument, we always measure  $V_1$  first and then  $V_2$ , the measured values of  $V_1/V_2$  will be systematically low.

Table 8.1. *Measured values of the diameter of a wire at various points along its length*

Length/m	Diameter/mm
0.0	1.259
0.0	1.263
0.0	1.259
0.0	1.261
0.0	1.258
0.1	1.252
0.2	1.234
0.3	1.209
0.4	1.214
0.5	1.225
0.6	1.248
0.7	1.258
0.8	1.256
0.9	1.233

## 8.6 Systematic variations

Look at the numbers in Table 8.1 which represent the measurements of the diameter  $d$  of a piece of wire at various points along its length  $x$ . If you were asked for the best value of the diameter and an estimate of the standard error in a single measurement, how would you proceed? (Stop at this point and decide.)

Let us see how our friends X and Y would tackle this problem. X is in no doubt. He has been told that the best value of a quantity is the mean of a set of measurements, and he has a calculator that calculates the mean and the standard error. So he happily feeds all the readings into his calculator and finds the mean, which is 1.245 mm, and the standard error, which is 0.018 mm.

Y, however, notices that the readings are not varying in a random manner and so he plots them on a graph, which is shown in Fig. 8.2. It is now obvious that the variation is systematic. He realizes that the mean of all the readings has no significance at all. The diameter was measured five times at  $x=0$ , so its value there is over-weighted. Accordingly, he replaces these five readings by the single number 1.260, which is their mean. He takes the mean of the ten values which he now has and obtains 1.239 as his best value.

Furthermore, he realizes that, since the diameter is undoubtedly varying along the length of the wire, the spread in the values over the whole range of  $x$  has nothing whatever to do with the standard error in a single measurement. In order to obtain the latter, he looks at the spread in the five values of  $x=0$  and obtains 0.002 mm as an estimate for  $\sigma$ . (Whether this is a random error or whether the cross-section is not circular at  $x=0$  we cannot say without being told more about the measurements.)

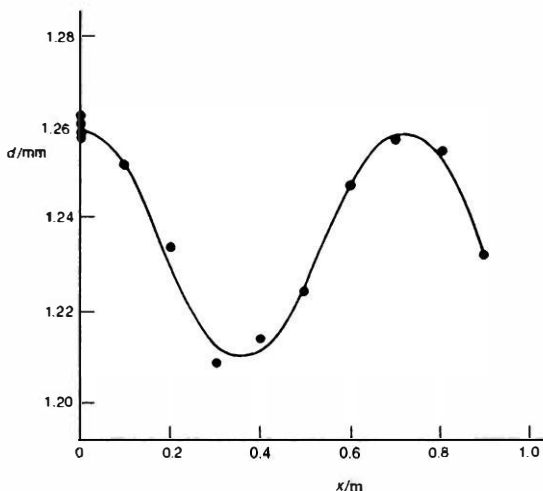


Fig. 8.2. Diameter of wire  $d$  at various points along its length  $x$  – plot of values in Table 8.1.

Y's approach is a sensible one, but there is a point to be noted in connection with the 'best value' of  $d$ . Since  $d$  is varying in a systematic way, the value we require is not necessarily  $d_m$ , the mean obtained by Y. If, for example, we have measured the resistance of the wire and wish to determine the resistivity of the material, the quantity required is the average value of  $1/d^2$ , which is not quite equal to  $1/d_m^2$ . In the present case the difference is small, but occasionally it is not, and the correct average must be taken.

Another situation that calls for examination is a set of results that spread by more than their errors indicate. Consider the set of results for the speed of sound in air at room temperature given in Table 8.2. We may suppose they were obtained by measuring the wavelength of stationary waves at various frequencies in a resonance tube. Suppose that at each frequency many measurements were made and that their internal consistency was such that the standard error in each result was estimated to be

$$\sigma = 0.7 \text{ m s}^{-1}. \quad (8.3)$$

In this situation some students simply take the mean of the five results and give as its error

$$\sigma_m = \frac{0.7}{\sqrt{5}} \approx 0.3 \text{ m s}^{-1}, \quad (8.4)$$

Table 8.2. *Measured values of the speed of sound*

Frequency/Hz	Speed/m s <sup>-1</sup>
1000	346.7
720	341.5
200	338.6
600	342.2
380	339.6

blithely ignoring the fact that three of the results are  $3\sigma$ ,  $4\sigma$ , and  $7\sigma$  away from the mean. If the value of  $\sigma$  given in (8.3) is reasonably correct, this is clear evidence of some systematic effect, and until it is discovered neither the mean nor the value of  $\sigma_m$  can be considered of much significance.

With all wave motion there is the possibility that the speed varies with frequency, a phenomenon known as *dispersion*. For sound waves in air, careful measurements by many experimenters have shown that there is no measurable dispersion at the frequencies in Table 8.2. However, in any resonance tube experiment there are certain corrections to be made, one of which depends on frequency (see Wood 1940, chapter X). It is possible that a systematic error in this correction has caused the present variation. Alternatively there might be a systematic error in the frequency values.

We therefore plot the values of the speed against frequency – Fig. 8.3. There does appear to be a correlation, and, if it is easy to make the measurements, it

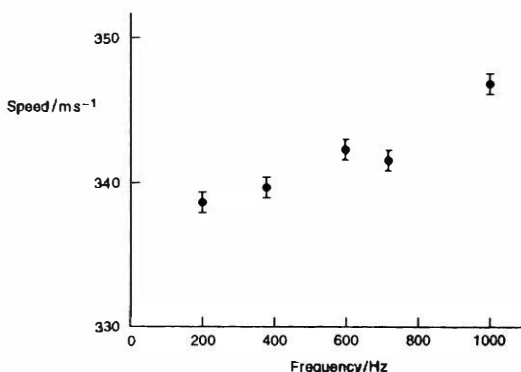


Fig. 8.3. Measured values of the speed of sound at various frequencies – plot of values in Table 8.2.



would be worth making a few more at other frequencies to see if the trend were confirmed. If so, we should look very carefully at the frequency-dependent correction and also at how the frequency values were obtained.

If the trend were not confirmed, we would have to look elsewhere. For example, although the value of  $\sigma$  in (8.3) might be correctly calculated from the measurements, it might be the case that the spread in the values at each frequency is spuriously small. This could arise if the resonance condition were being detected by ear, and a succession of measurements were made at the same frequency. The experimenter might have been influenced by the first reading at each frequency, and tended to find subsequent resonance conditions close to the first. This could be avoided by making only one or two measurements at each frequency and then repeating this cycle a few times without looking at the previous results.

We have treated this example in some detail to show possible ways of proceeding when results spread more than their apparent error suggests. It is a not uncommon situation in experimental work.

## 8.7 Calculated and empirical corrections

In many experiments, corrections have to be made to take account of systematic effects. In estimating the magnitude of these corrections, preference should always be given to empirical methods, i.e. methods based on actual measurements, rather than to theoretical calculations. The latter may be wrong for a variety of reasons – wrong theory, incorrect assumptions, faulty calculations – whereas empirical methods are, by their very nature, less likely to be in error.

Suppose, for example, we are investigating the transmission of light of a given wavelength through a certain liquid. We place the liquid in a glass cell with its end walls perpendicular to the light beam (Fig. 8.4) and measure the intensity  $I_X$  and  $I_Y$  of the light at the points X and Y. We suppose for simplicity that the transmission factor

$$f = \frac{I_Y}{I_X} \quad (8.5)$$

has been found to be independent of  $I_X$ . We require the value of  $f$  for the length  $l$

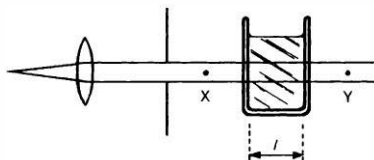


Fig. 8.4. Arrangement for measuring the attenuation of a light beam by a sample of liquid.

of the liquid alone. The cell cannot be entirely transparent, so it is necessary to correct the measurements for the attenuation of the light by the two end walls.

The theoretical way of making the correction would be to measure the thickness of the walls and look up in some table of physical constants the attenuation produced by this thickness of this particular glass for light of the wavelength we are using. Assuming that the information exists, the correction would depend on our knowing the correct wall thickness – it might not be constant and we would have to get the correct average value over just the area through which the light passed. It would depend on our knowing the wavelength of the light, and above all it would depend on our cell being made of just the same type of glass as that to which the tables referred.

The empirical method is first to measure the intensity at X and Y for the empty cell, then, without displacing the cell relative to the light beam, to fill the cell with the liquid and repeat the measurements. You can see that this procedure eliminates every one of the difficulties mentioned in the last paragraph.

Although empirical should be preferred to theoretical corrections, the best procedure of all is to obtain the corrections both ways and check that they agree. An example of this occurs in an experiment by Froome 1958, who measured the speed of light using microwave radiation (p. 116). The only part of the experiment that concerns us here is the measurement of the wavelength  $\lambda$  of the radiation. The principle of the measurement is shown in Fig. 8.5. The microwave signal from the source S is divided into two and applied to two transmitting horns  $T_1$  and  $T_2$ . The two signals are received by the horns  $R_1$  and  $R_2$  and combined in a unit not shown. The magnitude of the resultant signal depends on the relative phases of the signals received by  $R_1$  and  $R_2$ . If they are exactly in phase, it is a maximum; if they are exactly out of phase, it is zero. (Whatever the relative magnitudes of  $x_1$  and  $x_2$ , the signals received by  $R_1$  and  $R_2$  are made equal in amplitude. This is achieved by adjusting the relative amplitudes of the signals transmitted by  $T_1$  and  $T_2$  without altering their phases.)

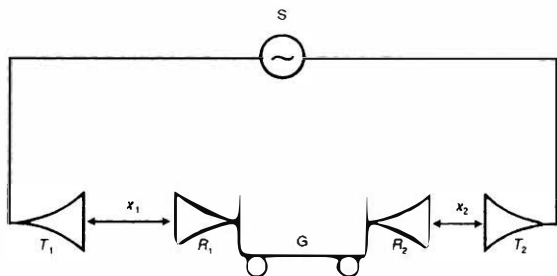


Fig. 8.5. Measurement of the wavelength in Froome's experiment.

$R_1$  and  $R_2$  are mounted on a rigid trolley  $G$  so that  $x_1$  and  $x_2$  can be varied, the quantity  $x_1 + x_2$  remaining constant. If the trolley is moved so that  $x_1$  increases by  $\lambda/2$ , then the phase of the signal at  $R_1$  is retarded by  $\pi$  and at  $R_2$  advanced by  $\pi$ . Thus the resultant signal goes through a complete cycle during the motion. The value of  $\lambda$  is obtained by counting the number of cycles through which the resultant signal passes as  $x_1$  or  $x_2$  is changed by a known amount.

Now the statement that increasing  $x_1$  by  $\lambda/2$  causes the phase of the signal at  $R_1$  to be retarded by  $\pi$ , is only strictly true in the limiting case where  $T_1$  is a point emitter and  $R_1$  a point receiver. In practice, radiation is emitted and received over a finite area. The path lengths of the rays from the various parts of  $T_1$  to the various parts of  $R_1$  vary. Moreover, when the value of  $x_1$  is changed, the different path lengths are changed by different amounts. The correction for these *diffraction* effects is important in a precision experiment such as this.

Fröome's procedure was to make measurements with various types of covering of the horns and at various values of  $x_1 + x_2$ . The diffraction effect, which varied, was calculated theoretically in each case. The agreement between theory and experiment throughout showed that in any given situation the correction was being calculated correctly and could be applied with confidence. This dual approach – theoretical and empirical – is the very acme of the experimental method.

## 8.8 Relative methods

The Wheatstone bridge, p. 103, is an example of a *relative* method. The resistance  $R$  is measured, not absolutely, but in terms of, or relative to, the resistance  $S$ . Relative methods are very important in physics. They can be made more precisely and easily than absolute measurements, and very often are all we require.

Consider as an example the measurement of the viscosity of a liquid by a method based on Poiseuille's formula for the flow of a liquid through a capillary tube. The formula is

$$\frac{dV}{dt} = \frac{p\pi r^4}{8l\eta}, \quad (8.6)$$

where  $dV/dt$  is the rate of volume flow of the liquid,  $p$  the pressure drop along a tube of length  $l$ ,  $r$  the internal radius of the capillary tube, and  $\eta$  the viscosity of the liquid. If we maintain and measure a constant pressure difference along the tube, and measure  $dV/dt$ ,  $l$ , and  $r$ , we can calculate the viscosity. This is an absolute measurement.

Now look at the apparatus in Fig. 8.6, which is a viscometer due to Ostwald. A fixed volume of liquid of density  $\rho_1$  and viscosity  $\eta_1$  is introduced into A and sucked into B so that its level is just above L on the left and N on the right. The time  $\tau_1$  taken by the liquid to drop from L to M – both levels being precisely marked – is measured. The liquid is then replaced by a second liquid of density  $\rho_2$

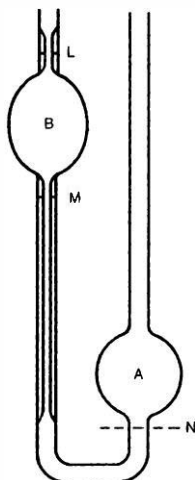


Fig. 8.6. Ostwald viscometer.

and viscosity  $\eta_2$ , and the corresponding time  $\tau_2$  is measured. It is easy to show that the ratio of the viscosities is given by\*

$$\frac{\eta_1}{\eta_2} = \frac{\rho_1 \tau_1}{\rho_2 \tau_2}. \quad (8.7)$$

The measurement of the quantities on the right-hand side of (8.7) is comparatively simple.

Notice that the relative method completely avoids two difficulties in the absolute method. The first is maintaining and measuring a constant pressure head, and the second is measuring the internal radius of the capillary tube, which must be done precisely since  $r$  is raised to the fourth power in the formula. Furthermore, the simplicity of the Ostwald apparatus means that temperature control is easier. As mentioned earlier, temperature has a marked effect on viscosity; so this is an important advantage.

The relative methods described so far give the *ratio* of two quantities, but a relative method can also give the *difference* between two quantities. A good example of this occurs in the measurement of  $g$ , the acceleration due to gravity. We have seen (p. 86) that to measure the absolute value of  $g$  with high precision an elaborate apparatus is necessary. However, it is possible to obtain precise

\* Equation (8.7), like Poiseuille's formula on which it is based, ignores the fact that the liquid acquires kinetic energy. A small correction is necessary to take account of this – see Smith 1960, chapter XI, for details.

values for differences in  $g$  with much simpler apparatus. The instrument used for the purpose is a *spring gravimeter*, which is essentially a very sensitive spring balance. There are several versions for use under different conditions. (See Dobrin and Savit 1988 for details.) The gravimeter must be calibrated by observations at a few places where the absolute value of  $g$  is already known. But it can measure differences in  $g$  with an error smaller than that of the absolute calibration values. The instrument is light, portable, and convenient to use, so that many observations can be made rapidly. Modern gravimeters have sensitivities of about  $100 \text{ nm s}^{-2}$ , which is comparable to the precision of the best absolute instruments.

For many purposes we are not interested in the absolute value of  $g$ , but only in its variation. The most important factors in the variation of  $g$  with position are latitude and height above sea-level – see Appendix G. When these have been allowed for, variations of  $g$  over large distances are related to the sea-level surface of the Earth and provide information about the structure of the continents and oceans; variations over small distances provide information of geological structure.

Other examples of relative methods are measurements of such quantities as the strength of a radioactive source, the intensity of a light source, and the flux density of radiation from a radio galaxy. In all these cases, the absolute determination of the quantity is very difficult, and the measurements are usually made relative to another similar quantity.

Notice finally that if we know the absolute value of the quantity we are measuring for *one* sample or object, all the relative values are converted into absolute values. Thus, once we have put one liquid of known viscosity and density through an Ostwald viscometer, measurements for another liquid give its viscosity absolutely.\* And once the value of  $g$  is determined absolutely at one of the points on the network of difference values, all these values become absolute.

## 8.9 Null methods

A *null* method is one in which the quantity  $X$  being measured is opposed by a similar quantity  $Y$ , whose magnitude is adjusted until some indicating device shows that a balance has been achieved. This is to be contrasted with a *direct* method in which the quantity being measured produces a deflection or reading on an instrument. We have already had two examples of null methods. The first is the compensator device in the Rayleigh refractometer, p. 75; the second is the Wheatstone bridge. Another example is the potentiometer where an unknown voltage is determined by balancing it against a known voltage.

Null methods have several important advantages over direct methods. The

\* The kinetic energy effect mentioned in the footnote on p. 112 requires that *two* liquids of known viscosity be put through the apparatus. But as you will already have realized – if you have caught on to the spirit of the last few chapters – the most satisfactory procedure is to calibrate the apparatus with *several* liquids of known viscosity.

adjusted quantity  $Y$  in a null method tends to be more stable and reproducible than the instrument in a direct method, and its value can usually be read more precisely. The instrument in a direct method must be calibrated and should preferably be a linear device. By contrast, the indicating device in a null method only has to show when some quantity is zero. Therefore it does not have to be calibrated and need not be linear, though it is convenient if it is so in the neighbourhood of zero. The main property of interest in the device is its sensitivity, that is, the minimum out-of-balance effect that it can detect.

A disadvantage of null methods is that they tend to be slower than direct ones, since the balance point is approached by stages. This can be overcome at the expense of further complication by using a servo system to find the balance point automatically. It is important to realize that null methods can give only relative, and not absolute, values. So an essential feature is the existence of a precisely known standard.

### 8.10 Why make precise measurements?

We said in chapter 2 that the precision to be sought in an experiment depends on its object. That is generally true, but in many experiments in physics, and particularly in measurements of fundamental quantities, we simply do not know what precision may, in the last resort, be appropriate. We try to attain the very highest precision that known phenomena and techniques permit. Why is this?

If you look at the table of values of the physical constants on p. 192, you will see that most of the constants are thought to be known to one part in  $10^7$  or even better. Now you may wonder if there is any point in this, or whether making such measurements is a useless exercise like calculating the value of  $\pi$  to hundreds of decimal places.

The answer is quite simple. Precise experiments have a very important purpose. They test our theoretical ideas and, when they give results at variance with them, lead to new theories and discoveries. There have been many examples of this in physics and chemistry. A theory says that two quantities are equal. We do an experiment and find that, within the limits of this experiment, they are. We then do a more precise experiment and find a small difference. In other words the theory is only a first approximation. The more precise experiment guides us in the next theoretical step. We conclude with a few examples of discoveries that have been made as a result of careful and precise measurements.\*

(a) Prior to 1894 it was thought that, apart from variable quantities of water vapour and traces of carbon dioxide, hydrogen, etc., atmospheric air consisted of oxygen and nitrogen. However, careful measurements by Rayleigh showed that

\* A detailed account of the measurement of the fundamental constants has been given by Petley (1985). An earlier article by Cohen and DuMond (1965) is well worth reading for its discussion and comments on the experimental method. For an instructive article on the importance of precise measurements in physics, see Cook 1975.

the density of the gas remaining when the oxygen had been removed was about  $\frac{1}{2}\%$  higher than the density of pure nitrogen obtained from a compound such as ammonia. This led Rayleigh and Ramsay (1895) to the discovery of the inert gas argon, which is now known to constitute about 1% of the atmosphere.

(b) The discovery of deuterium is another example of the fruitfulness of exact measurement. In 1929 the ratio of the mass of the hydrogen atom to the mass of  $^{16}\text{O}$  (the isotope of oxygen with mass number 16) was found by chemical determination of atomic weights to be

$$\frac{\text{mass H}}{\text{mass } ^{16}\text{O}} = \frac{1.00799 \pm 0.00002}{16} \quad (8.8)$$

In 1927 Aston had measured the ratio in his mass spectrograph and found

$$\frac{\text{mass H}}{\text{mass } ^{16}\text{O}} = \frac{1.00778 \pm 0.00005}{16} \quad (8.9)$$

The discrepancy between these two values led Birge and Menzel (1931) to suggest that what was being measured in the chemical determination was the *average* mass of the atoms in ordinary hydrogen gas, and if the latter contained a heavy isotope of mass 2, present in the proportion of 1 part in 5000, the discrepancy would be explained. (In the mass spectrograph only the light hydrogen atom contributes to the measurement.) The suggestion was confirmed soon after by Urey, Brickwedde, and Murphy (1932), who found faint lines in the spectrum of hydrogen. The wavelengths of these lines were in exact agreement with those calculated for the Balmer series of hydrogen with mass number 2.

(c) The constancy of the speed of light in empty space for all observers in uniform relative motion was first suggested by the experiments of Michelson and Morley between 1881 and 1887. They obtained interference fringes for light propagating in two directions at right angles and found no significant difference for the speed in the two directions, whatever time of the day or year the measurements were made. These and similar measurements led Einstein to the theory of special relativity, one of the great discoveries of physics. Even on pre-relativity theory the difference in the two speeds was expected to be small, and very precise measurements were necessary to show that the difference, if any, was very much less than the expected value. (For a clear account of the Michelson-Morley experiment, see Lipson, Lipson, and Tannhauser 1995.)

The original measurements of the speed of light  $c$  were done with optical waves for which the wavelength is about 500 nm. When measurements were made during and after the Second World War with microwaves – wavelength of the order of 10 mm – the value of  $c$  was found to be about  $17 \text{ km s}^{-1}$  higher than the optical value, despite the fact that the quoted standard errors in the two values were about  $1 \text{ km s}^{-1}$ . The difference is only 1 part in 20 000; nevertheless, had it been genuine the consequences for our present theories of electromagnetism would have been grave.

Table 8.3. Results of three experiments to measure the speed of light  $c$ 

Year	Experimenter	Range	$\lambda$	$c/\text{m s}^{-1}$
1958	Froome	microwave	4.2 mm	299 792 500 $\pm 100$
1972	Bay <i>et al.</i>	optical	630 nm	299 792 462 $\pm 18$
1973	Evenson <i>et al.</i>	infra-red	3.4 $\mu\text{m}$	299 792 457.4 $\pm 1.1$

Again, precise measurements were necessary to resolve the question. Repetition of the optical measurements did not confirm the previous values but gave results in agreement with the microwave values. The results of three representative experiments are given in Table 8.3. In each of the experiments the values of the frequency  $f$  and the wavelength  $\lambda$  of a monochromatic source were measured independently, and the value of  $c$  obtained from the relation  $c = f\lambda$ . The use of lasers (second and third experiments) give greatly improved precision. The experiment by Bay *et al.* was the first in which the very high frequency of an optical source (of the order of  $10^{14}$  Hz) was measured directly, instead of being deduced from a measurement of  $\lambda$  and the value of  $c$ . The procedure is to relate the optical frequency to a known microwave frequency by the generation of harmonics and the measurement of beat frequencies. A very readable account of the method has been given by Baird 1983; see also Udem *et al.* 1999.

The best evidence that the value of  $c$  is independent of wavelength comes from astronomical data. Radiation from pulsars and from high-energy gamma-ray bursts has been studied. From the times of arrival of pulses in different ranges of the electromagnetic spectrum, an upper limit may be calculated for the variation of  $c$  with wavelength  $\lambda$ . For radiation extending from microwaves ( $\lambda \approx 50$  mm) to gamma rays ( $\lambda \approx 2$  pm) the fractional variation  $\Delta c/c$  has been found to be less than  $10^{-12}$ . For gamma rays with wavelengths varying from about 40 pm to 6 pm,  $\Delta c/c$  has been found to be less than  $10^{-20}$ . See Schaefer 1999 for a summary of the results.



In the present chapter we are going to consider some common-sense aspects of doing experiments. They apply to all experiments, from the most elementary and simple to the most advanced and elaborate.

### 9.1 Preliminary experiment

In a real experiment, as opposed to an exercise, one nearly always does a trial experiment first. This serves several purposes.

(a) The experimenters 'learn' the experiment. Every experiment has its own techniques and routines, and the experimenter needs some training or practice in them. It is nearly always true that the first few measurements in an experiment are not as reliable or useful as the later ones, and it is usually more economical in time to have an initial period for finding out the best way of making the measurements and recording the results.

(b) The various pieces of apparatus are checked to see that they are working properly.

(c) A suitable range for each variable in the experiment is found.

(d) The errors in the different quantities can be estimated. As we have seen, this influences the strategy of the experiment proper, in the sense that more attention is given to those quantities whose errors give the major contributions to the final error.

Points (c) and (d) really add up to saying that any serious experiment must be planned, and that a few trial measurements provide a better basis for a plan than a lot of theory. Of course the plan must be flexible and is usually modified as the experiment goes along. But even the most rudimentary plan is preferable to making one measurement after another just as one thinks of them.

In an exercise experiment the scope for a preliminary experiment is somewhat limited, and you probably will not have time to do the whole experiment first, even roughly. Nevertheless, except in the very simplest experiment, *some* preliminary measurements should always be made and some sort of plan drawn up. This includes deciding what quantities are to be measured and roughly how long is to be spent on each one.

A related point may be made with regard to a piece of apparatus. Make sure that you know how it is to be operated, in the crude sense of knowing what

controls what, before you start making systematic measurements. If you are confronted with a spectrometer, for example, make sure before making *any* measurements, that you know how to rotate the prism table and how to rotate the telescope, which knob must be tightened to allow a micrometer screw adjustment to be effective, which vernier scale corresponds to which motion, and so on. If there is a laboratory manual giving information about the apparatus, or a leaflet put out by the manufacturer, read it *first*.

You may think this is all very obvious, and so it is. But it is surprising how many people are lacking in this elementary 'horse-sense' when it comes to experimental work. Sophisticated ways of treating data and avoiding subtle errors are all very well, but they are no substitute for common sense.

## 9.2 Checking the obvious

If the apparatus is meant to be mechanically firm, and most apparatus is, see that it is not wobbling about. Remember that three points define a plane, provided they are not in a straight line. So three is the best number of legs for a piece of apparatus, and the closer their positions are to an equilateral triangle the better. With more than three the apparatus will wobble when placed on a plane surface, unless the contact points have been made to lie in a plane.

If the base of the apparatus is meant to be level, and mostly it is, *look* and check that it is approximately so. You can always use a spirit-level afterwards if it must be accurately level.

In optical experiments, make sure that all reflecting and refracting surfaces look clean. A quick breath and wipe will often do wonders with cheap apparatus. But do *not* wipe expensive lenses with a cloth or handkerchief. They are made of soft glass, and in addition are often coated with a very thin film – about 100 nm thick – of a mineral salt to reduce reflection at the surface. Such lenses are easily scratched. They must never be touched with fingers and should be covered when not in use. They will seldom need more than a dusting with a soft camel-hair brush, or in extreme cases a careful wipe with special lens tissue.

See that optical components that are meant to be aligned, look as though they are, and that lenses are turned with their faces roughly at right angles to the direction of the beam. It is amazing how often one sees a student struggling with an optical system with some vital lens, covered with a film of grease, several millimetres too high or low and rotated  $10^\circ$  or so from the normal.

If you have to solder a connection in an electrical circuit, scrape the wires first, then make as firm a mechanical joint as possible. See that the solder melts in a healing flow running over the whole junction. Finally, when cold, wiggle the individual wires gently to make sure that the solder has adhered to them all and that you do not have a dry joint.

When you use a galvanometer or similar instrument with several ranges of sensitivity, always start with the switch on the least sensitive range.

When you are assembling electrical apparatus operated from the mains supply, always plug into the mains *last*, and if you have to service any of it, do not rely on the mains switch being off, but *pull the mains plug out*.

### 9.3 Personal errors

When you are making measurements you should regard yourself as a piece of apparatus subject to error like any other. You should try to be aware of your own particular errors. For example, in estimating tenths of a division in a scale reading some people have a tendency to avoid certain numbers. You can easily test yourself on this, which is perhaps not a very serious matter.

What can be serious, however, are what might be called wishful errors. Anyone is liable to make an occasional mistake in reading an instrument or in arithmetic. But suppose a series of measurements is producing results which you think are too high. You may well find yourself making more mistakes than usual, and the likelihood is that the majority will be in the direction of decreasing the result. Of course, if you do not know beforehand what to expect, you avoid the danger. This is not always possible, but can sometimes be achieved by a change of procedure or by varying the order in which measurements are made.

A related situation is one where measurements are repeated in rapid succession, and you are influenced by the previous values, so that if the first measurement is a misreading, you tend to repeat the mistake. Alternatively, even if there is no misreading, the results are not genuinely independent, and their spread is spuriously small – see p. 109.

In general you will make fewer mistakes if you are comfortable – physically and mentally. It is worth spending a little time, particularly in a lengthy set of measurements, to ensure this.

(a) Apparatus that needs to be adjusted and controls that are operated often should be conveniently placed.

(b) The same is true for instruments that are read often. In general it is more comfortable to read a vertical than a horizontal dial, and still more comfortable if the dial slopes backwards at a small angle.

(c) The general lighting should be good. (Of course in an optical experiment it is worth going to some trouble to *exclude* stray light.)

(d) Ventilation should be adequate. It is quite important that the air in the laboratory should be as fresh as possible – and not too warm.

(e) Finally, there should be somewhere convenient for your notebook, preferably away from sources of water and heat.

### 9.4 Repetition of measurements

Measurements of a single quantity should be repeated at least once. Such repetition

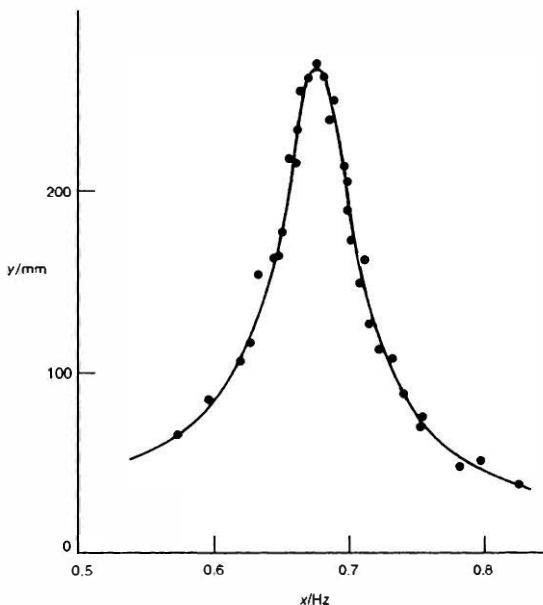


Fig. 9.1. Amplitude of oscillation of a simple harmonic system versus frequency of external force.

(a) serves to avoid mistakes in reading instruments and recording the numbers,

(b) provides a means of estimating the error.

But if we are making a series of measurements of a pair of quantities  $x, y$  from which we wish to determine say the slope  $m$  of the best line, there is no need to measure  $y$  several times for each value of  $x$ . Once we have made two pairs of measurements, i.e. obtained two points on the line, we have one value for  $m$ . We need further values for  $m$ , but it is better to obtain them by making more measurements at different values of  $x$ , rather than by repeating the measurements at the same values of  $x$ .

It is sometimes the case that we measure a set of  $x, y$  values and that the function  $y(x)$  is not at all like a straight line. Suppose for example that  $y$  is the amplitude of oscillation of a simple harmonic system due to an external oscillating force of frequency  $x$ . A typical set of  $x, y$  values in the neighbourhood of resonance is shown in Fig. 9.1. In this case the points must be sufficiently close

together to define the 'best' curve fairly well.\* But, as in the straight-line case, there is no need to repeat the measurements. The scatter of the points about the 'best' curve is a measure of their error, though it might be wise to repeat the measurement of  $y$  a few times for one or two values of  $x$  as a check.

There is one aspect to the repetition of measurements which may be illustrated by the following not-so-hypothetical occurrence. A student in a practical class comes to a demonstrator with a dilemma. He is measuring the angle of a prism with a spectrometer and has obtained the results

$56^{\circ} 30'$  and  $60^{\circ} 12'$ .

He estimates that his accuracy is about  $5'$  and, having checked the arithmetic, deduces that one of the results is wrong. (Stop at this point and consider what you would do.)

He asks the demonstrator which one he should take as correct. He is of course being ridiculous. The object of an experiment is to find out something. The student has so far found out nothing, except that one of his answers is wrong. (Quite possibly both are wrong.) In this situation *you must make more measurements*. In fact you must go on making measurements until the results start to make sense. You cannot do anything, not even the simplest manipulation, with the numbers until they *do* make sense.

If the next measurements gave  $56^{\circ} 34'$ , you would start to think that the second was probably wrong. You should measure the angle once more and, if the result was  $56^{\circ} 35'$ , you would be fairly certain that this was so. The results are beginning to be sensible. You may still wonder how you obtained the value  $60^{\circ} 12'$ . Probably you will never find out. Perhaps the prism was inadvertently moved during the measurements, or, more likely, a telescope setting was misread or incorrectly recorded. It is annoying when a wrong result turns up and you cannot account for it, but it does happen, and, provided it is rare, you need not worry. But you should worry if (a) you arbitrarily decided that the prism was a  $60^{\circ}$  one and therefore the second result was correct, or (b) you decided to take the mean.

## 9.5 Working out results

We say something about this in chapter 12, but here we only want to make the general point that in any experiment lasting more than a day or two, you should always try to work out the results as you go along.

It is very bad practice to make lots and lots of measurements and to work them out at the end of an experiment. First of all, it is much better to do the calculations while everything is fresh in your mind. Secondly, it is not uncommon

\* How close the points have to be to do this depends on the shape of the function and the precision of the measurements. Beginners are so used to measuring straight-line relations, where relatively few points are needed, that when they come to investigate relations such as that shown in Fig. 9.1, they usually do not have enough points.

to find when you work out a set of results that something is wrong and a change needs to be made to the apparatus. You will be much crosser if this happens after a month than after a day. Quite apart from that, it is often the case that one set of results determines what you do next.

The most foolish thing you can do in this respect is to dismantle elaborate apparatus before working out the results – it has been known.

### **9.6 Design of apparatus**

The principles and techniques of the design of apparatus are discussed in some of the books listed on p. 206. We say very little about the subject here, beyond offering a few words of general advice.

(a) Make things as simple as possible.

(b) If the apparatus is being constructed by a mechanic or instrument maker in the laboratory, discuss the detailed object of the apparatus with him before making definite plans or drawings. His experience may enable him to suggest improvements, or he may see ways of simplifying the apparatus and hence making it easier to construct, without detracting from its performance.

(c) Quote tolerances for the dimensions, and do not make them less than are actually necessary. Obviously, the closer the tolerances to which the constructor is asked to work, the more difficult is his task.

(d) Do some construction yourself. This will give you some idea of what can and cannot be done. An ounce of practice . . .

## Part 3

### Record and calculations





## 10 Record of the experiment

### 10.1 Introduction

In any experiment it is essential to keep a running record of everything that is done.

The record should be clear – and economical. On the one hand, you do not want to have to spend time subsequently searching pages of numbers without headings to find a particular set of results, or puzzling out from some meagre clues just what the conditions were when you made a certain set of measurements. On the other hand, to produce a record that is so clear that it may be followed with absolute ease by someone else is itself a time-consuming operation and is hardly necessary. You should aim at a record that you yourself will be able to interpret without too much difficulty after an interval of, say, a year.

In this chapter some suggestions for keeping the record are given. The important thing is not that you should regard them as a set of rules to be followed blindly, but rather that you should understand the spirit behind them, which is to produce a record – accurate, complete, and clear – with a minimum of effort.

### 10.2 Bound notebook versus loose-leaf

Some experimenters prefer a bound notebook; others use loose sheets. The advantage of a single bound book is that one knows where everything is – in the book. There are no loose bits of paper to be mislaid. The main disadvantage is that in an experiment of even moderate complexity one often goes from one part to another, and it is tiresome to have the various parts split into fragments in the record.

The advantage of loose-leaf is its flexibility. All the sheets on a particular topic may be brought together irrespective of what was written in between. The flexibility is useful in another way. In practical work it is convenient to use different kinds of paper – plain, ruled, graph, and data paper. (The last is paper ruled with vertical lines and is very convenient for tabular work.) You may also have sheets of computer output. All these can be inserted in any quantity and in any order in a loose-leaf book.

It is best not to be dogmatic about the basic method of keeping the record but to adapt the procedure to the particular experiment. A combination of bound

book and loose-leaf may secure the advantages of both. Whatever the system adopted, it is a good idea to have at least *one* bound notebook; it provides a place for odds and ends – occasional ideas, miscellaneous measurements, references to the literature, and so on. It is helpful if the pages of the notebook are numbered, and a detailed list of the contents compiled at the beginning or end of the book.

Experiments done by students as exercises are usually sufficiently short and straightforward to make a combination of bound notebook and loose-leaf sheets unnecessary. Opinion varies as to which is preferable, but the experience of the writer is that the flexibility of loose-leaf gives it the advantage. Different kinds of paper can be used in any order, and the account of a previous experiment may be handed in by the student for critical comment while he or she carries on with another.

### 10.3 Recording measurements

*All* measurements should be recorded *immediately* and *directly*. There is no exception to this rule. Do not perform any mental arithmetic – even the most trivial – on a reading before getting it down on paper. For example, suppose numbers appearing on an ammeter have to be divided by 2 to bring the readings to amperes. Write down the reading given by the instrument markings first. Do not divide by 2 and then write down the result. The reason for this is obvious. If you make a mistake in the mental arithmetic, you are never going to be able to correct it.

In making and recording a measurement, it is a good idea to check what you have written by looking at the instrument again. So

read, write, check.

Note the serial number of any instrument or of any key piece of apparatus, such as a standard resistor, used in the measurements. If the maker has not given it a readily observed serial number, you should give it one of your own. The subsequent identification of a particular piece of apparatus may be important. For example, something may go wrong in the experiment and the inquest may lead you to suspect an erratic instrument. You will want to know which particular one you used.

All written work should be dated.

### 10.4 Down with copying

An extremely bad habit of many students is to record the observations on scrap paper or in a 'rough' notebook and then to copy them into a 'neat' notebook, throwing away the originals. There are three objections to this:

- (a) It is a gross waste of time.
- (b) There is a possibility of a mistake in the copying.
- (c) It is almost impossible to avoid the temptation of being selective.

The last is the most important and is worth considering further. In most experiments we do not use all the measurements. We often decide that some measurements are not very useful, or were made under the wrong conditions or are simply not relevant. In other words, we are selective. This is quite proper, provided we have objective reasons for the selection. *But it is vital that all the original measurements be retained.* We may subsequently wish to make a different selection. And in any case all the experimental evidence must be available, so that someone else may form an opinion on the validity of our selection, or indeed on any aspect of the original measurements.

An important part of a practical physics course is to train you to keep clear and efficient records, but this training cannot begin until you try to make direct recordings. They will probably look very messy at first and perhaps will be difficult to follow, but do not let that discourage you. You will gradually learn from experience and improve. The result will never look as pretty as the copied-out version, but that is of no consequence. The important thing in a record is clarity, not beauty.

Having said that, let us add that a *certain amount* of copying out may well be useful. It is often a distinct aid to clarity, which in turn is desirable, not only for its own sake, but also because it reduces mistakes in working out the results. It is very often the case that at a certain point in the experiment we want to bring together various results dotted about in different parts of the account. We may want to plot a graph, do some calculations, or perhaps just look at the numbers. But, since we are retaining the original readings, this copying may be, in fact should be, highly selective and has nothing to do with the wholesale copying mentioned above.

## 10.5 Diagrams

'One picture is worth a thousand words' – Chinese proverb.

The importance of diagrams in a record or an account can hardly be exaggerated. Combined with a few words of explanation, a diagram is often the easiest and most effective way of explaining the principle of an experiment, describing the apparatus, and introducing notation. Consider the following alternative descriptions of a piece of apparatus for investigating the motion of a pair of coupled pendulums:

**Description 1.** A piece of string was fastened to a horizontal rod at two points A and B. Two spheres  $S_1$  and  $S_2$  were suspended by strings, the upper ends of which were attached to the original string by slip knots at the points  $P_1$  and  $P_2$ . The lengths AB,  $AP_1$ ,  $BP_2$ ,  $P_1P_2$  are denoted by  $a$ ,  $y_1$ ,  $y_2$ , and  $x$  respectively. The distance from  $P_1$  to the centre of  $S_1$  is denoted by  $l_1$ , and from  $P_2$  to the centre of  $S_2$  by  $l_2$ .

The degree of coupling between the pendulums was varied by varying the

distance  $x$ . This was done by moving the knots  $P_1$  and  $P_2$  along the string  $AP_1P_2B$ , the system being kept symmetric, i.e.  $y_1 = y_2$ .

**Description 2.** The apparatus was set up as shown in Fig. 10.1.  $AP_1P_2B$  is a continuous length of string.

$P_1$ ,  $P_2$  are slip knots.

Coupling varied by varying  $x$  by means of slip knots ( $y_1 = y_2$  throughout).

Comment on the two descriptions is superfluous.

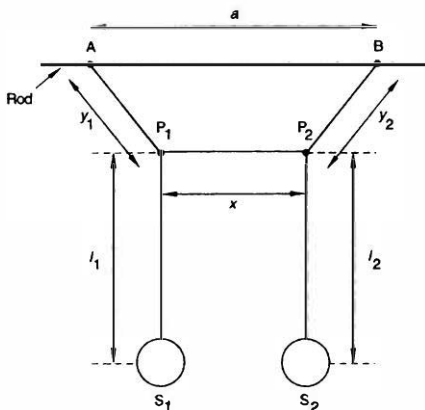


Fig. 10.1. Coupled pendulums.

A diagram should not be an artistic, or even a photographically true, representation of the apparatus. It should be schematic and as simple as possible, indicating only the features that are relevant to the experiment. Furthermore, although an overall sketch of the apparatus drawn roughly to scale is often helpful, you should not hesitate to distort the scale grossly in another diagram if this serves to make clear some particular point.

Suppose, to take a simple example, we are measuring the focal length of a convex lens by placing it on a plane mirror and observing when an object and its image are coincident. We wish to indicate whether a particular measurement refers to the distance from the object to the top or bottom of the lens. Figure 10.2a is to scale; Fig. 10.2b is not, but is much clearer for the purpose.

A diagram is usually the best way of giving a sign convention. Look at Fig. 10.3, which shows the usual convention for representing a rotation by a vector. Expressing this in words is not only more difficult but also less effective.

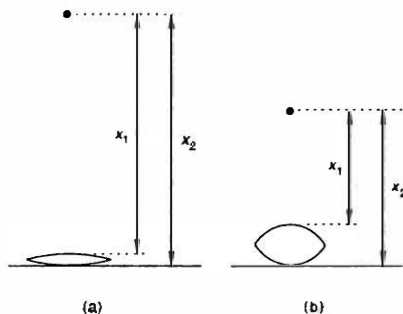


Fig. 10.2. Diagrams for coincident-image method of finding the focal length of a lens.

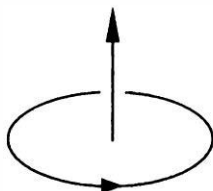


Fig. 10.3. Diagram showing the convention for representing a rotation by a vector.

## 10.6 Tables

When possible always record measurements in tabular form. This is both compact and easy to follow. Measurements of the same quantity should preferably be recorded vertically, because the eye can more readily compare a set of vertical numbers. Head each column with the name and/or symbol of the quantity, followed by the units.

For convenience, the power of 10 in the unit should be chosen so that the numbers recorded are roughly in the range 0.1 to 1000. There are several conventions for expressing units in a table. The one recommended by the Symbols Committee of the Royal Society\* – and used in this book – is that the expression at the head of a column should be a dimensionless quantity. Consider Table 10.1, which gives the values of  $\gamma$ , the surface tension of water, at various values of the temperature  $T$ . (The values are taken from Kaye and Laby 1995, p. 60.) We have

\* *Quantities, Units, and Symbols*. The Royal Society, 1975, 1981.

Table 10.1. *Surface tension of water*

$T/\text{K}$	$10^3 \text{ K}/T$	$\gamma/\text{mN m}^{-1}$
283	3.53	74.2
293	3.41	72.7
303	3.30	71.2

included also the values of  $1/T$  as a further illustration. The quantities  $T/\text{K}$ ,  $10^3 \text{ K}/T$ , and  $\gamma/\text{mN m}^{-1}$  are dimensionless, and are therefore suitable headings for the columns. The first line of numbers is interpreted thus.

$$\text{At } T/\text{K} = 283, \quad \text{i.e. } T = 283 \text{ K}, \quad (10.1)$$

the value of  $1/T$  is given by

$$10^3 \text{ K}/T = 3.53, \quad \text{i.e. } 1/T = 3.53 \times 10^{-3} \text{ K}^{-1}, \quad (10.2)$$

and at this temperature the value of  $\gamma$  is given by

$$\gamma/\text{mN m}^{-1} = 74.2, \quad \text{i.e. } \gamma = 74.2 \text{ mN m}^{-1}. \quad (10.3)$$

Once the unit has been specified at the top of a column it is not necessary to repeat it after each measurement. In general all unnecessary repetition should be avoided. It wastes time, it wastes energy, and it clutters up the record. The fewer inessentials we have, the easier it is to follow the essentials.

## 10.7 Aids to clarity

Diagrams and tables are two aids to clarity. But any such aid is to be welcomed. Groups of measurements of different quantities should be well separated and each should have a title. If a set of measurements leads to one value, say the mean, it is helpful if this one number is not only labelled but also underlined, or made to stand out in some way.

In general you should not be too economical with paper in keeping the record. You will often find yourself starting to record measurements without giving them a title or specifying the units. The habit of leaving a few lines of space at the top permits these items to be added tidily later on. Omitting titles to start with is not necessarily a sign of impatience, but is in fact a sensible practice. You will usually be able to add much more useful titles – often of an extended nature – after several sets of measurements have been made.

A definite hindrance to clarity is the habit of overwriting. Is 27, 27 or 37? Do not leave the reader – or yourself at a later date – to puzzle it out. Just cross out and rewrite 27 37.

## 10.8 Some common faults – ambiguity and vagueness

**Example 1.** A student is told to measure the viscosity of water at 20°C and to compare the value obtained with that given in some table of physical constants. The following appears in his notebook:

experimental value	$1.005 \times 10^{-3} \text{ N s m}^{-2}$
correct value	$1.002 \times 10^{-3} \text{ N s m}^{-2}$

Which is his value and which the one given in the tables? If we know him to be a modest person we might guess that what he calls the 'experimental value' is his own, and what he calls the 'correct value' is the one in the tables. If he is conceited it might be the other way round. But of course we ought not to have to guess, on the basis of his personality or otherwise. He should have written something like

this experiment	$1.005 \times 10^{-3} \text{ N s m}^{-2}$
Kaye & Laby (16th ed. p. 51)	$1.002 \times 10^{-3} \text{ N s m}^{-2}$

Other adjectives such as 'actual', 'official', 'measured', 'true' are equally vague in this context and should be avoided.

Note that a detailed reference to the tables has been given. We might want to check the value or look at the table again.

**Example 2.** Consider a notebook entry such as

Ammeter A 14 zero error  $-0.03 \text{ A}$

Does this mean that when no current passed the instrument read  $-0.03 \text{ A}$ , and therefore  $0.03$  should be *added* to all the instrument readings to get the correct value, or does it mean that  $0.03$  should be *subtracted*? Again we are left to guess.

In accordance with the rule that measurements should be written down directly, without the intervention of any mental arithmetic, what the experimenter should do is to read the instrument when no current is passing and write down the value. So the entry should be something like

Ammeter A 14  
 $-0.03 \text{ A}$  ← reading when no current passed

**Example 3.** The following is a statement of a kind often found in a student's notebook:

The timebase of the oscilloscope was checked with a timer-counter and found to be accurate within experimental error.

This is objectionable for its vagueness on two essential points. Firstly, an oscilloscope timebase has a number of ranges and the statement does not indicate which one or ones were checked. Secondly, the *evidence* for the statement has not

been given, and without that evidence we have no means of knowing whether the statement is justified. What should have appeared is something like this:

Oscilloscope SC 29 – calibration of timebase range  $0.1 \text{ ms cm}^{-1}$

sine-wave applied to Y plates  $f = 10.018 \text{ kHz}$  (timer-counter)

oscilloscope	cycle	$x/\text{cm}$	5 cycles
	0	0.95	5.00
	1	1.96	<u>4.92</u>
	5	5.95	$4.96 \pm 0.04 \text{ cm}$
	6	6.88	

$$\text{sweep speed} = \frac{5}{10.018 \times 4.96} = 0.1006 \pm 0.0008 \text{ ms cm}^{-1}$$

Conclusion timebase correct within experimental error for this range

The above examples illustrate the following points:

(a) What is stated must be unambiguous. You should consciously ask yourself whether any interpretation other than the correct one is possible. Quite often the simplest way to resolve possible ambiguity is to give a single numerical example.

(b) If a conclusion is based on numerical evidence, and nearly all conclusions in physics experiments are or should be so based, then *the numbers must be given explicitly*.



### 11.1 The use of graphs

In experimental physics, graphs have three main uses. The first is to determine the value of some quantity, usually the slope or the intercept of a straight line representing the relation between two variables. Although this use of graphs is often stressed in elementary teaching of practical physics, it is in fact a comparatively minor one. Whether we obtain the value of the slope of a straight line by the method of least squares or by taking the points in pairs (see chapter 4), we are of course not using the graph as such, but the original numbers. The only time we actually use the graph to determine the slope is when we judge or guess the best line through the points by eye. This is a crude method – though not to be despised on that account – and should only be used as a check on the result of a more sophisticated method, or when the value of the slope is not an important quantity in the final result.

The second use of graphs is by far the most important. They serve as *visual aids*. Suppose, for example, the rate of flow of water through a tube is measured as a function of the pressure gradient, with the object of determining when the flow ceases to be streamlined and becomes turbulent. A set of readings is listed in Table 11.1. (They are taken from Reynolds' original paper on turbulent flow – Reynolds 1883.) As long as the flow is streamlined, the rate of flow is proportional to the pressure difference. It is difficult to tell by inspecting the numbers in the table where the proportionality relationship breaks down. However, if the numbers are plotted on a graph (Fig. 11.1), the point of breakdown is apparent at once.

Another example of this visual use is the comparison of experimental results with a theoretical curve, when both are plotted on the same graph. In general a graphical display of the measurements is invaluable for showing what is going on in an experiment.

The third use of graphs in experimental work is to give an empirical relation between two quantities. For example, a thermometer may have been calibrated against some standard, and the error determined as a function of the thermometer reading (Fig. 11.2a). We draw a smooth or average curve through the measured errors and use it (Fig. 11.2b) to correct the thermometer readings. We could have done the same thing by compiling a correction table. In general a table is more convenient to use than a graph, but may be more trouble to compile.

Table 11.1. *Flow of water through a tube*

Pressure gradient/ $\text{Pa m}^{-1}$	Average velocity/ $\text{mm s}^{-1}$
7.8	35
15.6	65
23.4	78
31.3	126
39.0	142
46.9	171
54.7	194
62.6	226
78.3	245
86.0	258
87.6	258
93.9	271
101.6	277
109.6	284
118.0	290

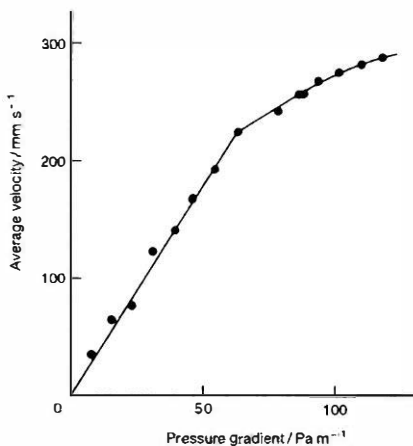


Fig. 11.1. Average velocity of water through a tube versus pressure gradient – plot of values in Table 11.1.

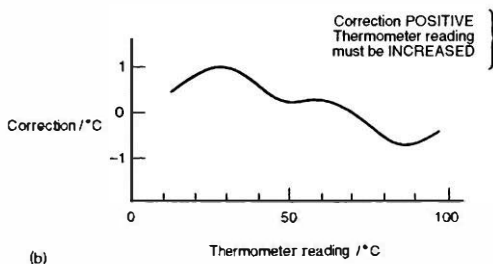
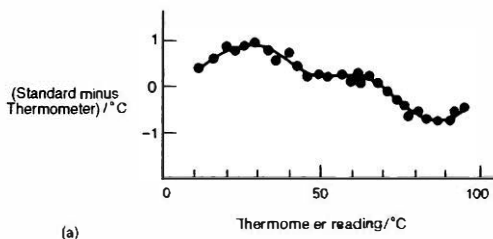


Fig. 11.2. (a) Calibration measurements for a thermometer and (b) correction curve.

It is a well-established convention for all graphs in physics that the independent variable, i.e. the quantity whose value is chosen each time by the experimenter, is plotted along the horizontal axis, and the dependent variable, i.e. the quantity whose value is then determined, plotted along the vertical axis. Or, more briefly, plot *cause* along the horizontal and *effect* along the vertical axis.

Nowadays there are a number of computer applications that plot graphs for you. The data consist of pairs of values for the two variables, sometimes with, in addition, error estimates for one or both of the variables. These are fed into the application, which then produces a graph. Most graph-plotting applications have a wide range of features. You can choose the axis scales, the symbols for the experimental points, and so on, and there are usually options to include error bars and curve fits. All this is very convenient, particularly if there is a large amount of data. However, you are advised to start by doing your own graph plotting in simple experiments with relatively few data. You will learn what are the most effective graphs to plot and will be in a better position to instruct a computer in more complicated experiments with large amounts of data.

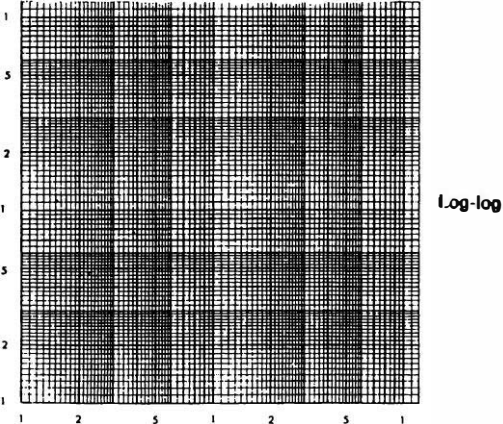
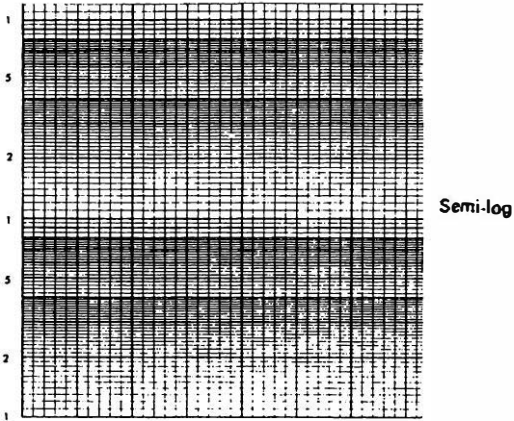


Fig. 11.3. Graph paper with logarithmic rulings.

## 11.2 Choice of ruling

Graph paper can be obtained in a variety of rulings for specialized purposes, but the two most commonly used in physics are ordinary linear and logarithmic rulings. The latter may be further subdivided according to whether only one axis is ruled logarithmically (semi-log) or both axes (log-log) – see Fig. 11.3. Semi-log paper is useful when there is a logarithmic or exponential relation between the two variables. Log-log paper is useful when the relation is of the form

$$y \propto x^p$$

and the value of  $p$  is not known. An alternative to the use of paper with logarithmic ruling is to take the logarithm of the quantity and plot the latter on a linear scale – see for example Fig. 4.3.

## 11.3 Scale

Suppose we are using graph paper ruled in centimetres and millimetres. Our choice of scale should be governed by the following considerations:

(a) The experimental points should not all be cramped together. It is rather difficult to get much out of Fig. 11.4a. So we choose a scale to make the points cover the sheet to a reasonable degree as in Fig. 11.4b. However, in trying to do this we should bear in mind two further points.

(b) The scale should be simple. The simplest is 1 cm on the graph paper representing 1 unit (or 10, 100, 0.1, etc.) of the measured quantity. The next simplest is 1 cm representing 2 or 5 units. Any other scale should be avoided simply because it is tedious to do the mental arithmetic each time a point is inserted or read off.

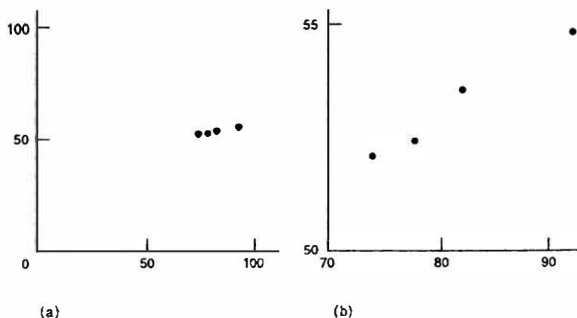


Fig. 11.4. (a) is not a very useful graph. The same results are plotted on an expanded scale in (b).

(c) We sometimes have to choose the scale for theoretical reasons. Thus if we are investigating whether the results in Fig. 11.4 satisfy the relation  $y = mx$ , we must include the origin in a plot of  $y$  against  $x$ , and Fig. 11.4b is wrong. (This does not mean that we need revert to Fig. 11.4a – see section 11.7.)

## 11.4 Units

It is usually convenient to choose the power of 10 in the unit on the same basis as for tables (see section 10.6). The marks on the graph can then be labelled 1, 2, 3, ... or 10, 20, 30, ... rather than 10 000, 20000, etc., or 0.0001, 0.0002, etc.

The axes should always be labelled with the name or symbol or both of the varying quantity. The units should be given with the same convention as in tables – see p. 129. Some examples are given in Fig. 11.5.

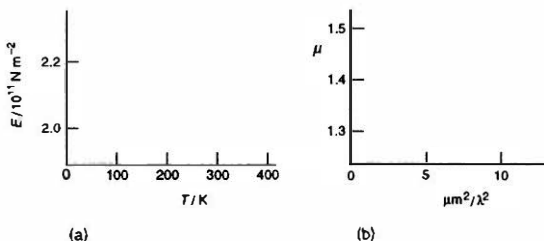


Fig. 11.5. Examples of labelling axes and expressing units. (a) Young modulus  $E$  versus temperature  $T$ . (b) Refractive index  $\mu$  of a glass versus  $1/\lambda^2$ , where  $\lambda$  is the wavelength of the light.

## 11.5 Some hints on drawing graphs

The main purpose of a graph is to give a visual impression of results, and it should therefore do this as clearly as possible. We give here a few general hints for drawing graphs. They should be interpreted and modified according to the particular case.

(a) If a theoretical curve is drawn on the graph for comparison with the experimental results, the calculated theoretical points through which the curve is drawn are chosen arbitrarily. They should therefore be inserted faintly, preferably in pencil, so that they can be rubbed out.

On the other hand, each experimental point should be represented by a bold mark – not a tiny point – so that it stands out clearly – Fig. 11.6.

(b) It sometimes helps if a ‘best’ smooth curve is drawn through the experi-

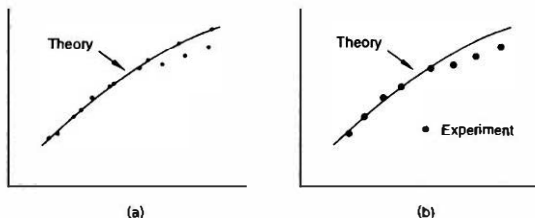


Fig. 11.6. (a) is a poor graph – the experimental points are faint and indistinguishable from the points calculated for drawing the theoretical curve. In (b) the calculated points have been erased, and the experimental points are prominent.

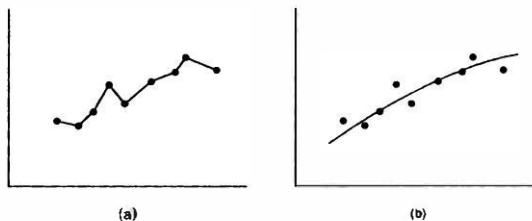


Fig. 11.7. (a) is wrong – it implies that the relation between the two variables has the jagged shape shown, which is most unlikely. From the experimental values we expect the relation to be something like the curve in (b).

mental points. Note the word *smooth*. Beginners sometimes join up experimental points as in Fig. 11.7a. But this implies that the relation between the two variables has the jagged shape shown, which except in special circumstances is highly unlikely. We expect the relation to be represented by something like the curve shown in Fig. 11.7b.

If a theoretical curve is also drawn on the graph, it is usually better to omit the curve through the experimental points. The curve may imply more than the actual results warrant and distracts from the direct comparison of experiment and theory.

(c) Different symbols, e.g. ●, ○, ×, or different colours may be used to distinguish experimental points that refer to different conditions or different substances. But they should be used sparingly. If the graph starts to look complicated it is better to draw each set on a separate graph.

The device of different symbols is perhaps most useful in demonstrating that varying the conditions or the material has little or no effect. An example is given

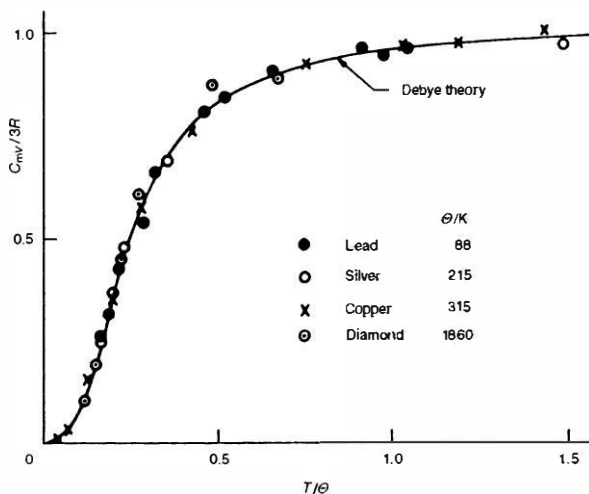


Fig. 11.8. Molar heat capacity,  $C_{mv}$ , in units of  $3R$ , versus  $T/\Theta$  for lead, silver, copper, and diamond.

in Fig. 11.8, where  $C_{mv}$ , the molar heat capacity at constant volume of a substance, is plotted against  $T/\Theta$ .  $T$  is the thermodynamic temperature, and  $\Theta$ , known as the Debye temperature, is a constant which depends on the substance. According to the Debye theory of specific heats, the relation between  $C_{mv}$  and  $T/\Theta$  is the same for all solids. Some results for lead ( $\Theta=88$  K), silver ( $\Theta=215$  K), copper ( $\Theta=315$  K) and diamond ( $\Theta=1860$  K) are given in the figure, together with the form of the relation predicted by Debye. It can be seen that for these substances the experimental results are in good agreement with the theory.

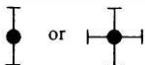
Notice that the quantity plotted along the  $y$ -axis is  $C_{mv}/3R$ , where  $R$  is the molar gas constant. It is a common procedure in physics to express a physical quantity in dimensionless form by means of some natural unit. In the present case the unit,  $3R$ , is the value of  $C_{mv}$  predicted by classical theory and also by the Debye theory in the high-temperature limit ( $T \gg \Theta$ ).

(d) It is a good idea to mark out the scale along the axes and to insert the experimental points in pencil in the first instance. You will sometimes change your mind about the scale and occasionally put a point in the wrong place initially. When you are satisfied with the scale and position of the points, it is easy to ink everything in and draw bold experimental points. The practice avoids messy alternations or wasting graph paper through redrawing.



## 11.6 Indicating errors

The error in an experimental point may be indicated thus



Since inserting the error bars is an additional labour and complicates the graph, it should only be done if the error information is thought to be relevant. Little would be gained for example by adding the error bars to the points in Figs. 9.1 or 11.8.

On the other hand, the significance of deviations from a theoretical curve may depend on the estimated error, and in that case the errors should be indicated. Thus in Fig. 11.9a the deviations would not be considered significant, whereas in Fig. 11.9b they would. We have already encountered this situation in section 8.6, where the spread of a set of results is not consistent with their errors. In that case the theoretical curve is the straight line

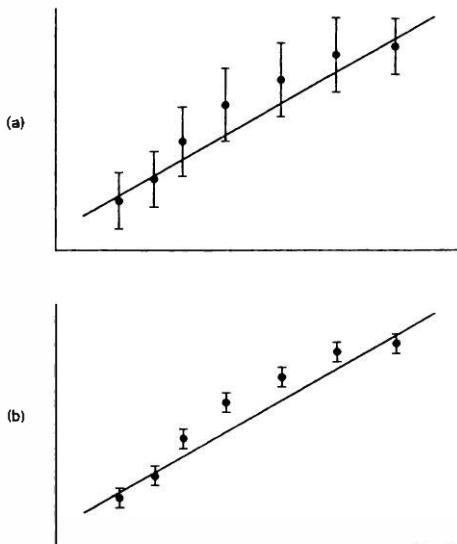


Fig. 11.9. The deviations are the same in the two figures, but in (a) they are probably not significant, whereas in (b) they probably are.

speed of sound = constant.

Plotting the experimental results together with the estimated error – Fig. 8.3 – is a useful way of showing the discrepancy.

Another situation in which errors are commonly shown is when they are different for the various experimental points.

## 11.7 Sensitivity

Suppose we are doing an experiment to determine whether a certain equation

$$y = x$$

is valid. We obtain pairs of values for  $y$  and  $x$  and find that the equation is approximately true. If we wish to show the results graphically, we can plot  $y$  against  $x$  – Fig. 11.10a. However, it is much more sensitive to plot  $y - x$  against  $x$ , because  $y - x$  is small compared to  $y$ , and we may use a much expanded scale for

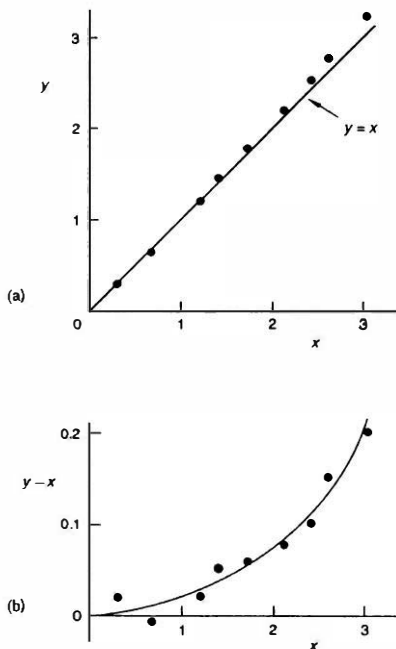


Fig. 11.10. (a)  $y$  versus  $x$ , and (b)  $y - x$  versus  $x$ .

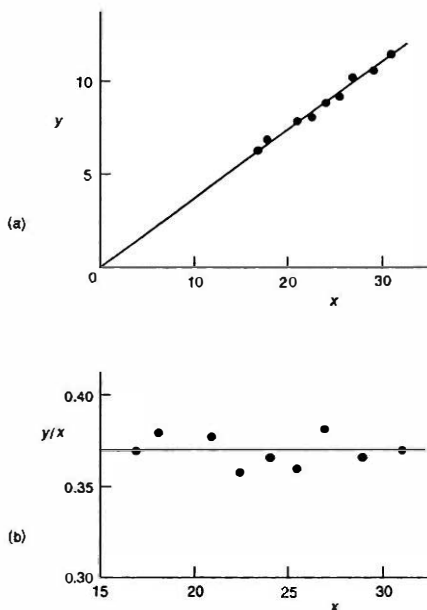


Fig. 11.11. (a)  $y$  versus  $x$ , and (b)  $y/x$  versus  $x$ .

it – Fig. 11.10b. The departure from the equation  $y = x$ , slightly indicated in the first figure, is quite evident in the second. Figure 11.2a is an example of this way of plotting results.

A similar method is applicable to the relation

$$y = mx.$$

A direct plot of  $y$  against  $x$  gives an overall picture of the relation and may be useful on that account – Fig. 11.11a. But plotting  $y/x$  against  $x$  is much more sensitive. We do not have to include the origin as in the direct plot, but may have whatever range of values for  $y/x$  and  $x$  is convenient – Fig. 11.11b.

### 12.1 Arithmetic is important

The object of an experiment is to obtain a number, and the correct working out of that number is just as important as the taking of the measurements. Many experiments performed by students, containing sensible measurements, are ruined by mistakes in calculating the results.

The following devices are available for calculations:

computer,  
calculator,  
you.

They are listed in order of decreasing expense and increasing availability. Choose the one appropriate to the job.

### 12.2 Computers

There are experiments where large computers are necessary, for example in the processing of radio and optical images studied in astrophysics, or in the determination of complicated biological structures by the analysis of X-ray diffraction patterns. However, for the type of experiment we are concerned with, the small computers available in the laboratory and in many homes are entirely adequate.

The great majority of the calculations that require a computer are best done with a spreadsheet. A spreadsheet such as Excel<sup>®</sup> is extremely versatile; it has a large range of functions – arithmetical, trigonometric, statistical, and logical – and a variety of features for controlling the appearance of the output, which includes the number of digits shown for a number. You can put comments and labels throughout the output, and you are strongly advised to do this liberally, so that when you look at it at a later date you will follow what has been done. You can, and should, save the result of a particular calculation as a document. This can serve as a template for a standard calculation. We have already had an example of this in the calculation of the best line through a set of points in a graph by the method of least squares – see the comment at the end of section 4.2 (d), p. 36.

### 12.3 Calculators

Although a computer is the most efficient tool for complicated or repetitive calculations, there are many calculations in an experiment which are sufficiently simple for a calculator to be the most appropriate device for the purpose.

A calculator, being portable and versatile, is an indispensable tool for doing arithmetic in practical work. All calculators do the four basic operations of addition, subtraction, multiplication, and division. It is worth getting what is termed a *scientific calculator*, which also calculates squares, square roots, and the common functions – logarithmic, exponential, and trigonometric. A number of storage registers, or memories is also an advantage. Some calculators have built-in programs for calculating the mean and standard deviation of a set of numbers, and the best straight line through a set of points. A calculator that you can program yourself is an advantage for repetitive calculations, but these are best done with a computer if one is available.

Most calculators produce numbers with up to 8 or 10 digits. This is of course far more than is significant for the vast majority of experiments. You should avoid the elementary blunder of quoting values with a large number of meaningless digits just because they are given by the calculator. Many calculators allow you to fix the number of digits after the decimal point. This facility is very convenient and should certainly be used if available. It is much easier to appreciate the significance of numbers, and you will make fewer mistakes, if you eliminate most of the insignificant digits.

You should always retain at least one and possibly two digits beyond those that are significant. You should never so truncate a number that you lose information. So if you are not sure at the time how many digits *are* significant, err on the side of caution and include more rather than less.

### 12.4 Ways of reducing arithmetical mistakes

It might be thought that with calculators and computers there is no need to worry about arithmetical mistakes. Experience suggests otherwise. It is true that calculators and computers are very reliable. They will nearly always give the right answer based on what you tell them. The trouble is that what you tell them may be wrong. You may enter a wrong number, or press a wrong function key, or make a mistake in program logic. Anyone is liable to make such mistakes. However, it is possible to reduce their likelihood by sensible procedures. Moreover, there is a remedy, namely, to *check* the calculation. We consider these points in turn.

(a) *Avoid unnecessary calculations.* The less calculating you do, the less opportunity you have for going wrong, and the more mental energy you have available for necessary calculations.

Suppose, for example, you are doing a simple experiment to determine the force constant  $\lambda$  of a spring, defined by the equation

$$F = \lambda x, \quad (12.1)$$

where  $F$  is the applied force in newtons and  $x$  the resulting extension in metres. You have a set of weights 1, 2, 3, . . . , 6 kg and you measure the extension produced by each one in turn. Do you then convert each weight into newtons by multiplying by 9.81? You would be very foolish to do so. That would involve six multiplications and, moreover, would convert the six simple integers into six messy numbers. You should of course do all the arithmetic to find the best value of  $\lambda$ , keeping  $F$  in the integer form. Only at the end should you multiply the value of  $\lambda$ , in units of kilogram-weight per metre, by 9.81 to get the answer in newtons per metre.

The same point applies when a quantity is measured a number of times with an instrument which has been calibrated against a standard. The calibration correction should be made only to the mean of the measurements, and not to each measurement separately.

**(b) Be tidy.** Calculations should be set out as systematically and tidily as possible. Space the working liberally. Untidy and cramped calculations are a prolific source of mistakes.

Most of the points made in connection with recording measurements apply also to calculations. A tabular arrangement of numbers is often the most convenient and effective. (Of course, if you are doing the calculation on a computer with a spreadsheet this arrangement is automatic.) Very often the numbers in one column are the results of manipulations of numbers in one or more previous columns. Every column should have a heading of some kind to indicate the manipulation. These headings are often simplified if the columns are in addition labelled alphabetically. For example, suppose you are calculating a function that occurs in resonance theory

$$y = \frac{1}{Q} \left[ (1-x)^2 + \frac{x}{Q^2} \right]^{-\frac{1}{2}} \quad (12.2)$$

for the value  $Q = 22$ . A simple set of table headings would be

A	B	C	D	E	F
$x$	$(1-x)^2$	$\frac{x}{484}$	$B+C$	$1/\sqrt{D}$	$y = \frac{E}{22}$

**(c) Check the calculation.** Checking should be regarded as part of the calculation. The experimenter is here in the same situation as a manufacturer of motor cars. The latter must have a department for inspecting the cars before they are sent out. This is regarded as a necessary part of the cost of producing a car. In the same way, part of your time and effort must be devoted to checking the calculations.

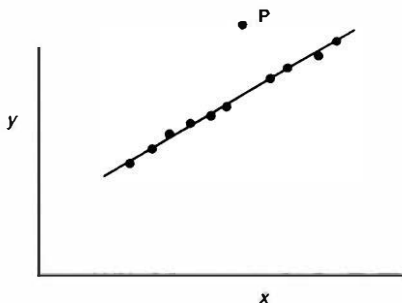


Fig. 12.1. We need not check the arithmetic leading to each point very carefully, because a mistake will probably be obvious – P is almost certainly a mistake.

But it is up to you to get the greatest return for the effort, in other words, to direct it where it is most needed. Some calculations in an experiment are more important than others and should therefore be checked more carefully.

Calculations can be divided into two categories, which may be termed 'self-checking' and 'non-self-checking'. Suppose for example we are measuring two physical quantities which after a little arithmetic give a pair of numbers  $x_i$ ,  $y_i$ . The measurements are plotted in a graph and give approximately a straight line. We would be justified here in not checking the arithmetic very carefully for every  $x_i$  and  $y_i$ , because if we made a mistake it would very likely stand out on the graph – Fig. 12.1. This is an example of a set of self-checking calculations.

But suppose we end an experiment with the quantity to be determined given by

$$Z = \frac{14.93 \times 9.81 \times 873}{6.85 \times (0.7156)^2 \times \pi^2}. \quad (12.3)$$

The evaluation of this expression is not self-checking. There is nothing to tell us whether the answer is correct, except some method of checking. If a calculator is being used, it is worth doing the calculation twice, particularly if the numbers in (12.3) are not already stored in the calculator memories and have to be keyed in.

You may think it is showing excessive caution to work out the result twice as suggested; but arithmetical mistakes in experiments are a major source of wasted effort. On balance you will save time by careful calculation. Remember that, though in a classroom someone may go through your work and find your mistakes, no one will do so later on. You should get into the habit of making a rough mental estimate – good to about 1 in 3 – of every calculation. Thus for the expression in (12.3) you should say something like

$$\frac{14.93}{6.85} \approx 2,$$

$$2 \times 9.81 \times 873 \approx 20000,$$

$$(0.7156)^2 \times \pi^2 \approx \frac{1}{2} \times 10 = 5.$$

So

$$Z \approx 4000. \quad (12.4)$$

So far we have assumed that the numbers in (12.3) are correct, i.e. free from calculation errors. But of course this may not be so, particularly if they are themselves the end results of long calculations. One of the conveniences of a calculator is that it is not necessary to record the results of intermediate steps, and it is tempting not to do so. However, the consequence is that if you make a mistake somewhere along the way it is not possible to trace and correct it. If the repetition of a long calculation gives a different answer you still do not know which one (if either) is correct. It is therefore important to record some of the intermediate values. How many and which ones depend on the complexity of the calculation and your own reliability. You would be well advised to record more rather than less at the beginning, and to reduce the amount with experience.

If a check calculation does not agree with the original, look at the check first, as it was probably done less carefully than the original. A story is told of a new research student in theoretical physics who took the result of an elaborate calculation to his supervisor, a very famous physicist. The supervisor looked at it and said, 'If we take the following special case, your result should reduce to such and such.' He scribbled a couple of lines of calculation on the back of an envelope and said 'You see, it doesn't. You've gone wrong somewhere.' The downcast student took his work away and spend the next month going through it all again. He sought his supervisor once more. 'Well,' said the great man, 'did you find the mistake?' 'Yes,' replied the student, 'it was in your two lines of calculation.'

Look at all numerical values to see if they are reasonable. If you divide instead of multiply by the Planck constant in SI units, you should notice that the answer looks wrong. Of course, to know whether a value is reasonable or not you must have some idea of the orders of magnitude of various physical quantities in the first place – see exercise 12.1.

## 12.5 Checking algebra

(a) **Dimensions.** The dimensions of an algebraic expression provide a useful check. It is not worth checking the dimensions at every stage of a theoretical argument, but an expression such as

$$l^2 + l,$$

where  $l$  is a length, should leap to the eye as being wrong.



In expressions like  $\exp x$ ,  $\sin x$ ,  $\cos x$ , the dimensions of  $x$  must be zero. This is true of any function that can be expressed as a power series in  $x$ .

(b) *Special cases.* You should check that an expression reduces to the correct form in special simple cases – the very famous physicist had the right idea.

(c) *Direction of variation.* Make sure that an expression gives a variation in the right *direction* as some component quantity varies. Consider, for example, Poiseuille's equation

$$\frac{dV}{dt} = \frac{p\pi r^4}{8l\eta}. \quad (12.5)$$

(The quantities are defined on p. 111.) We expect on physical grounds that if  $p$  or  $r$  is increased, or if  $l$  or  $\eta$  is decreased, then  $dV/dt$  should increase. The form of the equation is in agreement with each of these variations.

(d) *Symmetry.* Symmetry sometimes provides a useful check on the correctness of a formula. The resistance of the arrangement in Fig. 12.2 is

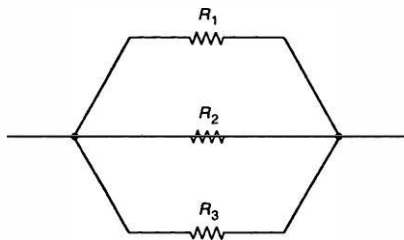


Fig 12.2. Arrangement of resistors.

$$\frac{R_1 R_2 R_3}{R_1 R_2 + R_2 R_3 + R_3 R_1}. \quad (12.6)$$

If we exchange any two of the  $R$ s, say  $R_1$  and  $R_2$ , we get the same expression, which of course we must, because the arrangement is symmetric in the three resistances. But suppose we had made a mistake in the algebra and had obtained the result

$$\frac{R_1 R_2 R_3}{R_1 (R_2 + R_3)}. \quad (12.7)$$

We would know at once that it was wrong, because interchanging  $R_1$  and  $R_2$  gives a different result.

Table 12.1. *Approximations for some functions for  $x \ll 1$*

Function	Approximation
$(1+x)^{\frac{1}{2}}$	$1 + \frac{1}{2}x$
$\frac{1}{1+x}$	$1 - x$
$(1+x)^a$	$1 + ax$
$\sin x$	$x$
$\cos x$	$1 - \frac{1}{2}x^2$
$\tan x$	$x$
$\exp x$	$1 + x$
$\ln(1+x)$	$x$

(e) **Series expansion.** The first one or two terms of a series expansion of a function often provides a useful approximation. Some common functions are given in Table 12.1.

(f) **A useful tip.** If you have to do some algebraic manipulation of quantities for which you have numerical values, do all the algebra first and get the final algebraic answer. Substitute the numerical value right at the end. It is much easier to avoid mistakes in this way. Quite apart from that, once you substitute numbers you lose the check on dimensions.

## Exercises

**12.1** The following exercises are given to test your knowledge of orders of magnitude of various physical quantities. They are in the form of simple problems, because there is not much point in knowing the value of a physical quantity unless you know how to make use of it. You should try the problems in the first instance without looking up any of the values. Try to make an intelligent guess of those you do not know, on the basis of any theoretical or practical knowledge you have of related quantities.

When you have gone through all the problems in this way, look up the values in a book like Kaye and Laby before looking at the answers. You will find browsing through Kaye and Laby far more instructive than just checking the answers given here.

- A copper rod is 40 mm in diameter and 200 mm long. If one end is at  $0^\circ\text{C}$ , how much heat must be supplied per second at the other end to keep it at  $25^\circ\text{C}$ ?
- A steel rule is correct at  $20^\circ\text{C}$ . What is its error at  $30^\circ\text{C}$ ?

- (c) A copper wire of diameter 1 mm is 1 m long. (i) What is its resistance at  $0^{\circ}\text{C}$  and (ii) by how much does it change between  $0^{\circ}\text{C}$  and  $20^{\circ}\text{C}$ ?
- (d) The cold junction of a chromel-alumel thermocouple is at  $0^{\circ}\text{C}$  and the hot junction is at  $100^{\circ}\text{C}$ . What emf results?
- (e) Water flows through a tube 1 mm in diameter and 250 mm long under a pressure difference of  $2000 \text{ Nm}^{-2}$ . What is its average velocity (i) at  $20^{\circ}\text{C}$  and (ii) at  $50^{\circ}\text{C}$ ?
- (f) The cross-section of a rectangular steel bar is 25 mm by 5 mm. If its length is 1 m, what force is necessary to extend it by 0.5 mm?
- (g) What is the wavelength in air at  $0^{\circ}\text{C}$  of a sound wave of frequency 256 Hz?
- (h) What is the average (rms) velocity of a hydrogen molecule at  $27^{\circ}\text{C}$ ?
- (i) Estimate a value of  $G$  from  $g$ .
- (j) Parallel monochromatic red light is incident normally on a diffraction grating. (i) If the first order spectrum is at an angle of  $30^{\circ}$  to the normal, estimate the number of lines per mm of the grating. What roughly would be the angle for (ii) green light and (iii) violet light?
- (k) How much energy is radiated per second by a black sphere of radius 20 mm at a temperature of 500 K?
- (l) What is (i) the velocity and (ii) the wavelength of an electron whose energy is 1 keV?
- (m) What magnetic field is required to bend a stream of protons of energy 1 MeV into circle of radius 500 mm?
- (n) What is the wavelength of light that will just ionize a hydrogen atom in the ground state?
- (o) What is the energy equivalent in MeV of a mass of unit atomic weight?

## 12.2 Calculate the following in your head:

- (a)  $1.00025 \times 1.00041 \times 0.99987$ ,
- (b)  $912.64 \times \left( \frac{7200.0}{7200.9} \right)^2$ ,
- (c)  $(9.100)^{\frac{1}{2}}$ .

### 13.1 Introduction

The communication of ideas, theories, and experimental results is an important part of scientific work. Vast quantities of scientific literature are pouring out into the world, and if you take up a scientific career of any kind you are almost certain to add to the flood. If you can achieve a good standard of writing, two benefits will accrue – one to yourself when people take note of what you have to say, and the other to the rest of the world who – strange to say – prefer their reading matter to be clear and interesting rather than obscure and dull.

We are going to consider some elementary features of good scientific writing in the present chapter. To make the discussion specific we shall confine it to a paper on some experimental work in physics, but much of what we have to say applies to scientific writing in general.

### 13.2 Title

The title serves to identify the paper. It should be brief – not more than about 10 words. You should bear in mind that the title will ultimately appear in a subject index. The compiler of an index relies heavily on the words in the title in deciding where it should appear. So if there are one or two key words which help to classify the work, try to put them in the title.

### 13.3 Abstract

Every paper should have an abstract of about 100 words or so, giving positive information about its contents.

The abstract serves two classes of reader. It enables those who work in the subject to decide whether they want to read the paper; and it provides a summary for those who have only a general interest in the subject – they can obtain the essential results without having to read the whole paper. The abstract should therefore not only indicate the general scope of the paper but should contain the final numerical results and the main conclusions.

### 13.4 Plan of paper

Most papers – unless they are very short – are divided into sections. The following is a fairly common division:

Introduction  
Experimental method  
Results  
Discussion

Some papers describing experimental work also contain theoretical material which might well constitute an additional section, coming after either Introduction or Results.

Although the actual plan of a paper depends to some extent on its contents, you can see that the one above is logical and you should try to follow it, at least in a general way. We consider each section in turn.

### 13.5 Sections of paper

**(a) Introduction.** The Introduction is an important part of the paper. Most experiments are part of a general investigation of a physical problem. The Introduction should make clear

- (i) the physical interest of the problem,
- (ii) the part played by the experiment in the general investigation,
- (iii) the relation of the experiment to any previous work.

These points add up to your answering the question ‘Why did you do the experiment or what was its *object*?’

You may assume that the reader of the main body of the paper has a certain background knowledge of the subject, but it may be that someone starting the paper does not have this knowledge. The Introduction should serve as a possible starting point for him or her. You may not wish to go back to the beginning of the subject, in which case you should give references – not too many – to published work which does provide the necessary background. The Introduction plus this work should bring the reader to the point where he or she is ready to hear about your experiment.

We give an example of a splendid opening. It is from J. J. Thomson’s paper on Cathode Rays (Thomson 1897), announcing the discovery of the electron.

#### CATHODE RAYS

The experiments discussed in this paper were undertaken in the hope of gaining some information as to the nature of the Cathode Rays. The most diverse opinions are held as to these rays; according to the almost unanimous opinion of German physicists they are due to some process in

the aether to which – inasmuch as in a uniform magnetic field their course is circular and not rectilinear – no phenomenon hitherto observed is analogous; another view of these rays is that, so far from being wholly aetherial, they are in fact wholly material, and that they mark the paths of particles of matter charged with negative electricity. It would seem at first sight that it ought not to be difficult to discriminate between views so different, yet experience shows that this is not the case, as amongst the physicists who have most deeply studied the subject can be found supporters of either theory.

The electrified-particle theory has for purposes of research a great advantage over the aetherial theory, since it is definite and its consequences can be predicted; with the aetherial theory it is impossible to predict what will happen under any given circumstances, as on this theory we are dealing with hitherto unobserved phenomena in the aether, of whose laws we are ignorant.

The following experiments were made to test some of the consequences of the electrified-particle theory.

We shall have more to say about this passage later. But in the meantime, notice how clearly and directly Thomson has given the kind of information that should appear in the Introduction – the opening sentence is a model of its kind.

**(b) *Experimental method.*** In this section comes the description of the apparatus. The amount of detail here varies considerably, and you must use your own judgement, but a few general principles may serve as a guide.

If the apparatus you used is of a standard kind, it is probably sufficient to say what it was and give a reference so that anyone interested can find a full description. On the other hand, if it contains some novel features, they should be described in some detail. If the paper is intended for a journal devoted to descriptions of instruments and apparatus, still more detail would be appropriate. But we shall suppose that this is not the case and that the main interest is in the results and their interpretation, rather than on the apparatus used.

Although you may assume that the reader of this section has a certain familiarity with the background to the work, you should not go farther than this. You should certainly not aim the paper directly at other experimenters using the same or similar apparatus. So you should not use esoteric phrases understood only by such workers. Nor should you include minute experimental details of interest only to them.

**(c) *Results.*** In general it is neither possible nor desirable to give all the measurements. They would only confuse and distract the reader. He or she would have to spend time assessing their relative importance and extracting the essential results. But that is your job before writing the paper.

You should, therefore, give only

- (i) a representative sample of some of the basic measurements,
- (ii) the important results.

Note the word *representative* in (i). The sample that you present in the paper should give a faithful picture of the quality, precision, and reproducibility of the measurements. So if you have fifty sets of them, you do not reproduce the second best with the caption 'Typical set'.

**(d) Discussion.** The heading speaks for itself. Like the Introduction, this section is an important part of the paper. It should include

- (i) comparison with other similar measurements, if there are any,
- (ii) comparison with relevant theories,
- (iii) discussion of the state of the problem under investigation in the light of your results. This is the *conclusion*, the counterpart to the *object* of the experiment, given in the Introduction.

### 13.6 Diagrams, graphs, and tables

Almost everything said in chapters 10 and 11 about diagrams, graphs, and tables applies to their use in papers.

A diagram can be a great help in understanding the text. Unless the apparatus is completely standard, a diagram of it should nearly always be included. Graphs are a very convenient and common way of displaying the results. They should be kept simple; the same applies to diagrams. In the final printed version, graphs and diagrams are usually reduced in scale by a factor of two or three, so unless they are bold and clear in the original, they will look fussy and complicated in the final version.

Tables are a very good way of presenting results. They have the big advantage that they stand out, and the reader can find the results easily.

### 13.7 Instructions to authors

Most scientific journals now allow papers to be submitted either electronically or as hard copy. You will usually find instructions about this at the beginning of each issue. In addition, journals usually produce a pamphlet giving instructions to authors so that their papers may conform to the general style of the journal. You are advised to read this *before* the paper is prepared in its final form; otherwise someone – probably you – will have to spend a lot of time bringing it into line later on.

The pamphlet gives instructions on the form of sectional headings, abbreviations, references, footnotes, tables, diagrams, and graphs. It also gives instructions on the arrangement of mathematical material. This is important, because the way

you write a mathematical expression in longhand may be inconvenient or expensive to set up in print. For example, most journals prefer

$$(a^2 + b^2)^{1/2} \quad \text{to} \quad \sqrt{a^2 + b^2}.$$

If the journal does not produce a pamphlet, you should examine a recent issue.

### 13.8 Clarity

Clarity is an essential quality in scientific writing. We may distinguish two kinds.

**(a) Structural clarity.** Writing may be said to have structural clarity when the reader can readily follow the outline of the argument – or see the wood despite the trees. Similar topics are grouped together, and the groups arranged in a logical order.

You are strongly advised to construct a framework before writing the paper. This is a skeleton outline in which all the ideas, arguments, experimental details and so on are represented by a word or phrase. When the items are in this form, the arrangement is seen much more clearly and, moreover, is easily changed if not satisfactory. The main sections of the scheme should correspond to the plan given in section 13.4.

**(b) Expository clarity.** The other type of clarity, which may be called expository clarity, is making the reader understand exactly what you are trying to say at each stage in the discussion.

Look at the extract from the paper on Cathode Rays again. It is crystal clear. We are led firmly from one point to the next. Notice the phrase ‘so far from being wholly aetherial’. It could have been omitted, and we would still have followed the discussion; but the explicit contrast is helpful. Making it easy for the reader is a worthwhile object in any writing, but particularly in scientific writing.

Perhaps you feel that the example we have given is not a very severe test for the writer, because he is explaining something simple. That is true, but the reason it is simple is that Thomson has made it so. He has *selected* the important features of the theories of the nature of the rays. He is able to do this because he *understands the physics*. And this is a fundamental point. Clear writing depends on clear thinking. You will not be able to produce a clear and logical paper unless you do understand the physics.

### 13.9 Good English

We come to the final link in the chain between you and the reader – the words themselves. Good English in scientific writing is not just a matter of correct grammar – though that is not unimportant – it is choosing words and composing sentences to say exactly what you mean as concisely and pleasantly as possible.



Some instructive books on the subject are given on p. 206. We make a few miscellaneous points here.

(a) Students are often discouraged from using 'I' in their accounts. There seems no sensible reason for this. When you are describing an experiment you actually did, the 'I' style is a natural one and enables you to use the active voice, which is simpler and more direct than the passive. However, it must be admitted that nowadays few papers, even those describing experiments, are written in the first person. So if you want to be conventional, avoid it. But if you do use it, you will be in the company of Newton, Faraday, Maxwell, and Thomson, which, as Damon Runyon might have said, is by no means bad company at that.

(b) On the whole, short sentences make for clarity, but variety is necessary to avoid monotony. You can be clear in long sentences – Thomson's are hardly short – but it takes more skill.

(c) Paragraphs can help the reader to follow the argument. Start a new paragraph when you are starting a fresh point, or when you start discussing a point from a different angle.

(d) Avoid verbiage, roundabout ways of saying things, and redundant adverbs. Thus the second version should be preferred to the first in the following examples:

- (i) Calculations were carried out on the basis of a comparatively rough approximation.
- (ii) Approximate calculations were made.
- (i) Similar considerations may be applied in the case of copper with a view to testing to what extent the theory is capable of providing a correct estimate of the elastic properties of this metal.
- (ii) The elastic properties of copper may be calculated in the same way as a further test of the theory.

(e) Avoid qualifying a noun with a long string of adjectives – most of which are not adjectives anyhow. For example

The time-of-flight inelastic thermal neutron scattering apparatus . . . should be replaced by

The time-of-flight apparatus for the inelastic scattering of thermal neutrons . . .

(f) The unattached participle is a common fault in scientific writing. Sentences like

Inserting equation (3), the expression becomes . . .

or

Using a multimeter, the voltage was found to be . . .

occur frequently, so much so that you may not even realize what is wrong with them. Perhaps the following example taken from Fowler (1965) will show. A firm wrote to a customer 'Dear Sir, Being desirous of clearing our books, will you please send a cheque to settle your account. Yours etc.' They received the

following reply: 'Sirs, you have been misinformed. I have no desire to clear your books.'

### 13.10 Conclusion

Not everyone can write great literature, but anyone can write good, clear English – if he or she is prepared to take the trouble. Be critical of what you write. Ask yourself continually whether what you have written is logical, clear, concise. If not, try again – and again. Hard writing makes easy reading. Give your work to others to read for critical comment; read and criticize theirs.

You should not regard good experimenting and good writing as separate things. There is beauty in both, and it is no accident that the greatest scientists like Galileo and Newton have produced some of the finest scientific writing. But we shall allow a non-scientist – Cervantes – the final word.

Study to explain your Thoughts and set them in the truest Light, labouring as much as possible not to leave them dark nor intricate, but clear and intelligible.

## Appendices



## Appendix A

### Evaluation of some integrals connected with the Gaussian function

$$\text{A.1 } \int_{-\infty}^{\infty} \exp(-x^2) dx$$

First consider the integral with finite limits. We put

$$U = \int_{-a}^a \exp(-x^2) dx = \int_{-a}^a \exp(-y^2) dy, \quad (\text{A.1})$$

since the variable is a dummy. Therefore

$$U^2 = \int_{-a}^a \exp(-x^2) dx \int_{-a}^a \exp(-y^2) dy. \quad (\text{A.2})$$

Since one integrand is a function only of  $x$  and the other only of  $y$ , the right-hand side is equal to  $\exp[-(x^2 + y^2)]$  integrated over the square ABCD in Fig. A.1, i.e.

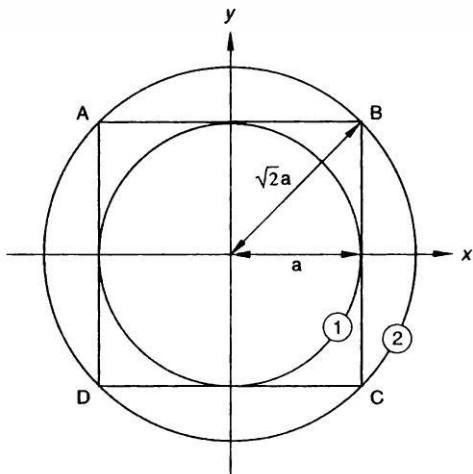


Fig. A.1. Evaluation of the Gaussian integral.

$$U^2 = \int_{-a}^a \int_{-a}^a \exp[-(x^2 + y^2)] dx dy. \quad (\text{A.3})$$

We now change to polar coordinates  $r, \theta$ . The area element  $dx dy$  is replaced by  $r dr d\theta$ , and  $\exp[-(x^2 + y^2)]$  becomes  $\exp(-r^2)$ .

Consider the integral of  $\exp(-r^2)$  over a circle of radius  $b$ .

$$W(b) = \int_0^b \int_0^{2\pi} \exp(-r^2) r dr d\theta = \pi[1 - \exp(-b^2)]. \quad (\text{A.4})$$

Clearly  $U^2$  is greater than  $W(b)$  for  $b = a$  (circle ① in Fig. A.1) but less than  $W(b)$  for  $b = \sqrt{2}a$  (circle ②); i.e.  $U^2$  lies between

$$\pi[1 - \exp(-a^2)] \quad \text{and} \quad \pi[1 - \exp(-2a^2)].$$

But as  $a$  tends to infinity, both these limits tend to  $\pi$ , and therefore  $U^2$  also tends to  $\pi$ . So

$$\int_{-\infty}^{\infty} \exp(-x^2) dx = \sqrt{\pi}. \quad (\text{A.5})$$

A.2  $\int_{-\infty}^{\infty} \exp(-x^2/2\sigma^2) dx$

This is evaluated from (A.5) by changing the variable to  $y = x/\sqrt{2}\sigma$ .

$$\begin{aligned} \int_{-\infty}^{\infty} \exp(-x^2/2\sigma^2) dx &= \sqrt{2}\sigma \int_{-\infty}^{\infty} \exp(-y^2) dy \\ &= \sqrt{(2\pi)}\sigma. \end{aligned} \quad (\text{A.6})$$

A.3  $I_n = \int_0^{\infty} x^n \exp(-x^2/2\sigma^2) dx$

$n$  is a positive integer or zero. We first evaluate  $I_0$  and  $I_1$ .

$$I_0 = \int_0^{\infty} \exp(-x^2/2\sigma^2) dx = \sqrt{\left(\frac{\pi}{2}\right)}\sigma. \quad (\text{A.7})$$

This follows from (A.6). The integrand has the same value for  $-x$  and  $x$ . So the integral from 0 to  $\infty$  is half the value of the integral from  $-\infty$  to  $\infty$ .

The integral  $I_1$  is evaluated by direct integration.

$$\begin{aligned} I_1 &= \int_0^{\infty} x \exp(-x^2/2\sigma^2) dx = 2\sigma^2 \int_0^{\infty} y \exp(-y^2) dy \\ &= \sigma^2. \end{aligned} \quad (\text{A.8})$$

Having evaluated  $I_0$  and  $I_1$  we may evaluate  $I_n$  for general values of  $n$  by treating  $\sigma$  as a variable parameter and differentiating under the integral sign.

$$\frac{d}{d\sigma} \int_0^\infty x^n \exp(-x^2/2\sigma^2) dx = \frac{1}{\sigma^3} \int_0^\infty x^{n+2} \exp(-x^2/2\sigma^2) dx, \quad (\text{A.9})$$

$$\text{i.e.} \quad I_{n+2} = \sigma^3 \frac{dI_n}{d\sigma}. \quad (\text{A.10})$$

So for  $n > 1$

$$I_n = 1 \cdot 3 \cdot 5 \cdot \dots \cdot (n-1) \sqrt{\left(\frac{\pi}{2}\right)} \sigma^{n+1} \quad n \text{ even}, \quad (\text{A.11})$$

$$I_n = 2 \cdot 4 \cdot 6 \cdot \dots \cdot (n-1) \sigma^{n+1} \quad n \text{ odd}. \quad (\text{A.12})$$

If the integration is taken from  $-\infty$  to  $\infty$ , the values of the integral are doubled for even  $n$ . For odd  $n$ , the integral is zero, since the integrand is an odd function of  $x$ .

## Appendix B

### The variance of $s^2$ for a Gaussian distribution

We prove the result, stated in section 3.7, that for a Gaussian distribution the variance of  $s^2$  is  $2\langle s^2 \rangle^2 / (n-1)$ .

For one set of  $n$  readings

$$s^2 = \frac{1}{n} \sum d_i^2. \quad (\text{B.1})$$

The error in  $s^2$  is

$$u = s^2 - \langle s^2 \rangle. \quad (\text{B.2})$$

The quantity we require is

$$\begin{aligned} \langle u^2 \rangle &= \langle s^4 - 2s^2 \langle s^2 \rangle + \langle s^2 \rangle^2 \rangle \\ &= \langle s^4 \rangle - \langle s^2 \rangle^2. \end{aligned} \quad (\text{B.3})$$

From (3.8) and (3.16)

$$\begin{aligned} s^2 &= \frac{1}{n} \sum e_i^2 - \frac{1}{n^2} (\sum e_i)^2 \\ &= \left( \frac{1}{n} - \frac{1}{n^2} \right) \sum e_i^2 - \frac{1}{n^2} \sum_i \sum_{j \neq i} e_i e_j. \end{aligned} \quad (\text{B.4})$$

Square both sides and average over the distribution. The average of any term containing an odd power of  $e$  is zero. Therefore

$$\langle s^2 \rangle = \left( 1 - \frac{1}{n} \right) \langle e^2 \rangle, \quad (\text{B.5})$$

and

$$\langle s^4 \rangle = \left( \frac{1}{n} - \frac{1}{n^2} \right)^2 n \langle e^4 \rangle + \left( \frac{1}{n} - \frac{1}{n^2} \right)^2 n(n-1) \langle e^2 \rangle^2 + \frac{1}{n^4} 2n(n-1) \langle e^2 \rangle^2. \quad (\text{B.6})$$

From (B.3), (B.5), and (B.6)

$$\frac{\langle u^2 \rangle}{\langle s^2 \rangle^2} = \frac{1}{n} \left[ \frac{\langle e^4 \rangle}{\langle e^2 \rangle^2} - \frac{n-3}{n-1} \right]. \quad (\text{B.7})$$



For a Gaussian distribution we have from Table 3.3

$$\langle e^4 \rangle = 3\sigma^4 \quad \text{and} \quad \langle e^2 \rangle = \sigma^2, \quad (\text{B.8})$$

whence

$$\langle u^2 \rangle = \frac{2}{n-1} \langle s^2 \rangle^2. \quad (\text{B.9})$$

## Appendix C

### The straight line – the standard error in the slope and intercept

In section 4.2 we used the method of least squares to calculate the best values of  $m$  and  $c$  in the equation  $y = mx + c$ , for a set of measurements consisting of  $n$  pairs  $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$  with equal weights. We here derive the expressions for the standard errors  $\Delta m$  and  $\Delta c$  given in formulae (4.31) and (4.32).

Imagine that we keep repeating the measurements so that we have many sets each of  $n$  pairs, the measurements being made in such a way that the values  $x_1, x_2, \dots, x_n$  are the same for all the sets. In other words, if we look at the first pair  $x_1, y_1$  in all the sets,  $x_1$  is the same throughout, but the value of  $y_1$  varies. We shall have a distribution of  $y_1$ s centred about  $Y_1$ , the true value of  $y$  for  $x_1$ . Similarly for all the second pairs;  $x_2$  is the same in each case, but the  $y_2$ s vary and form a distribution centred on  $Y_2$ , the true value of  $y$  for  $x_2$ . And so on for all the pairs. Since the  $n$  pairs have equal weights, the standard deviations of the  $n$  distributions are equal and we denote them by  $\sigma$ . The situation is shown schematically in Fig. C.1. We assume that for each set there is no correlation between the errors in two different  $y$ s.

Now the  $x_i, Y_i$  values are related by

$$Y_i = Mx_i + C. \quad (\text{C.1})$$

This is the true line, and  $M$  and  $C$  are the true values of the slope and intercept.

For each set of  $n(x_i, y_i)$  pairs, we can calculate the values of  $m$  and  $c$  from the expressions given in (4.25) and (4.26). The value of  $m$  averaged over all the sets is  $M$ , and the standard error in a single value is  $\Delta m$ , where

$$(\Delta m)^2 = \langle (m - M)^2 \rangle, \quad (\text{C.2})$$

the average again being taken over all the sets. Similarly the average value of  $c$  is  $C$ , and the standard error of  $c$  is  $\Delta c$ , where

$$(\Delta c)^2 = \langle (c - C)^2 \rangle. \quad (\text{C.3})$$

In an actual experiment we only have one set of  $n(x_i, y_i)$  values. The values for  $m$  and  $c$  for this particular set are our best estimates of  $M$  and  $C$ . The problem is to obtain estimates for  $\Delta m$  and  $\Delta c$ .

The algebra is considerably simplified if we change the independent variable from  $x$  to  $\xi$ , given by

$$\xi = x - \bar{x}, \quad (\text{C.4})$$

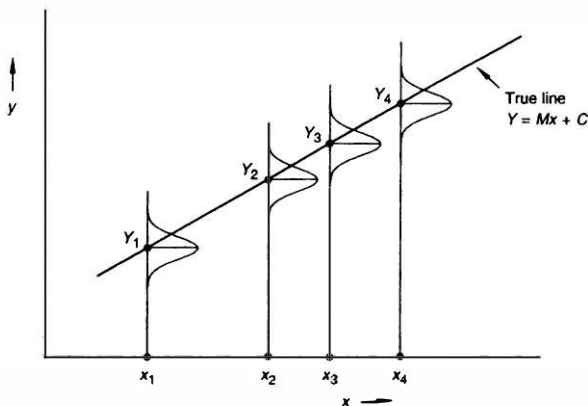


Fig. C.1. Repetition of measurements with a fixed set of  $x$  values. For each  $x$  the values of  $y$  form a distribution centred about the true value  $Y$ . Since the measurements at different  $x$  values have equal weights, the standard deviations of the distributions are equal.

where

$$\bar{x} = \frac{1}{n} \sum x_i. \quad (\text{C.5})$$

Clearly

$$\sum \xi_i = \sum (x_i - \bar{x}) = 0. \quad (\text{C.6})$$

The quantity  $D$  is defined by

$$D = \sum \xi_i^2 = \sum (x_i - \bar{x})^2 = \sum x_i^2 - n\bar{x}^2. \quad (\text{C.7})$$

The line

$$y = mx + c \quad (\text{C.8})$$

becomes

$$y = m(\xi + \bar{x}) + c \quad (\text{C.9})$$

$$= m\xi + b, \quad (\text{C.10})$$

where

$$b = m\bar{x} + c. \quad (\text{C.11})$$

The best values of  $m$  and  $b$  for a given set of  $n$  pairs of measurements are obtained from (4.25) and (4.26) with  $c$  replaced by  $b$ , and  $x$  by  $\xi$ . Since  $\sum \xi_i = 0$ , these equations become

$$m = \frac{1}{D} \sum \xi_i y_i, \quad b = \bar{y} = \frac{1}{n} \sum y_i. \quad (\text{C.12})$$

From now on the symbols  $m$  and  $b$  refer to these values.

We see that  $m$  is a linear function of the  $y_i$ . For a given set

$$m = \frac{\xi_1}{D} y_1 + \frac{\xi_2}{D} y_2 + \dots \quad (\text{C.13})$$

The coefficients of the  $y$ s are the same for all the sets. Since we are assuming that in each set there is no correlation between the errors in two different  $y$ s, we may use (4.17) and (4.18) to calculate  $\Delta m$  in terms of the errors in the  $y$ s.

$$(\Delta m)^2 = \left(\frac{\xi_1}{D}\right)^2 (\Delta y_1)^2 + \left(\frac{\xi_2}{D}\right)^2 (\Delta y_2)^2 + \dots \quad (\text{C.14})$$

But

$$(\Delta y_1)^2 = (\Delta y_2)^2 = \dots = \sigma^2. \quad (\text{C.15})$$

So we have

$$(\Delta m)^2 = \frac{\sum \xi_i^2}{D^2} \sigma^2 = \frac{\sigma^2}{D}. \quad (\text{C.16})$$

Similarly,

$$(\Delta b)^2 = \frac{1}{n} \sigma^2. \quad (\text{C.17})$$

We actually require  $(\Delta c)^2$ , which, from (C.11), is given by

$$(\Delta c)^2 = (\Delta b)^2 + \bar{x}^2 (\Delta m)^2 \quad (\text{C.18})$$

$$= \left(\frac{1}{n} + \frac{\bar{x}^2}{D}\right) \sigma^2. \quad (\text{C.19})$$

(See comment at the end of the Appendix.)

The estimate of  $\sigma$  is obtained as follows. If  $B$  is the true value of  $b$ , then

$$Y_i = M \xi_i + B. \quad (\text{C.20})$$

Adding these equations for each  $i$  gives an expression for  $B$  (since  $\sum \xi_i = 0$ ). Similarly, multiplying each one by  $\xi_i$  and then adding gives an expression for  $M$ . Thus

$$M = \frac{1}{D} \sum \xi_i Y_i, \quad B = \frac{1}{n} \sum Y_i. \quad (\text{C.21})$$

The error in the  $i$ th  $y$  reading is

$$e_i = y_i - Y_i = y_i - (M \xi_i + B). \quad (\text{C.22})$$

At the point  $\xi_i$ , the best line gives  $m \xi_i + b$  for the value of  $y$ . The residual  $d_i$  is therefore

$$d_i = y_i - (m \xi_i + b) \quad (\text{C.23})$$

~ Fig. C.2. As in the case of a single variable, the errors  $e_i$  are not known, but the

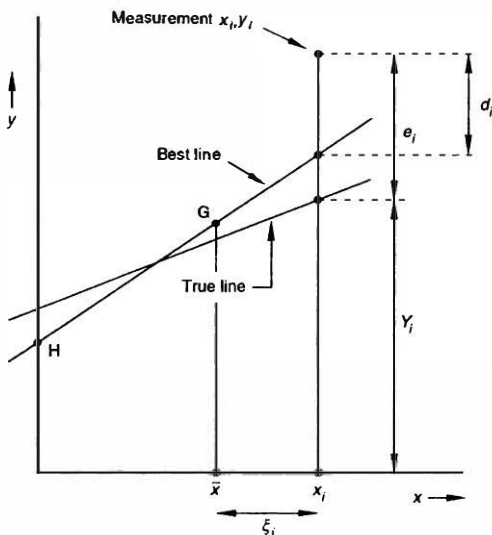


Fig. C.2. Diagram showing various quantities defined in Appendix C.

residuals  $d_i$  are known. The root-mean-square value of  $d_i$  for the  $n$  points is denoted by  $s$  as before.

From (C.22) and (C.23)

$$d_i = e_i - [(m - M)\xi_i + (b - B)]. \quad (\text{C.24})$$

From (C.12) and (C.21)

$$m - M = \frac{1}{D} \sum \xi_i (y_i - Y_i) = \frac{1}{D} \sum \xi_i e_i, \quad (\text{C.25})$$

$$b - B = \frac{1}{n} \sum e_i. \quad (\text{C.26})$$

Insert these expressions for  $m - M$  and  $b - B$  in (C.24), square both sides, and sum over  $i$ . This gives

$$\sum d_i^2 = \sum e_i^2 - \frac{1}{D} (\sum \xi_i e_i)^2 - \frac{1}{n} (\sum e_i)^2. \quad (\text{C.27})$$

(In summing over  $i$  we have again made use of the fact that  $\sum \xi_i = 0$ .)

Now average (C.27) over all the sets, remembering that the  $\xi_i$  are fixed, and that the average value of  $e_i e_j$  for  $i \neq j$  is zero. The average of the middle term on the right-hand side is

$$\frac{1}{D} \left\langle \left( \sum \xi_i e_i \right)^2 \right\rangle = \frac{1}{D} \left\langle \sum \xi_i^2 e_i^2 \right\rangle = \sigma^2. \quad (\text{C.28})$$

Equation (C.27) therefore becomes

$$n \langle s^2 \rangle = n \sigma^2 - \sigma^2 - \sigma^2, \quad (\text{C.29})$$

or

$$\sigma^2 = \frac{n}{n-2} \langle s^2 \rangle. \quad (\text{C.30})$$

Our best value of  $\langle s^2 \rangle$  is  $(1/n) \sum d_i^2$ . From (C.16), (C.19), and (C.30) we have the required results

$$\langle \Delta m \rangle^2 \approx \frac{1}{D} \frac{\sum d_i^2}{n-2}, \quad (\text{C.31})$$

$$\langle \Delta c \rangle^2 \approx \left( \frac{1}{n} + \frac{\bar{x}^2}{D} \right) \frac{\sum d_i^2}{n-2}. \quad (\text{C.32})$$

The generalization to the case of unequal weights is readily made. If the  $i$ th point has weight  $w_i$ , the variance of the  $i$ th distribution is put equal to  $\sigma^2(\sum w_i)/nw_i$ , where  $\sigma$  is a constant. The results given on p. 40 then follow by reasoning closely similar to that above.

### Comment on the dependence of $m$ , $c$ , and $b$

Equation (C.18) assumes that the values of  $m$  and  $b$  are independent, which may readily be proved by calculating  $\langle (m - M)(b - B) \rangle$  from (C.12) and (C.21). The average value of this quantity is seen to be zero. However,  $m$  and  $c$  are *not* independent, for

$$\langle (m - M)(c - C) \rangle = -\bar{x}(\Delta m)^2. \quad (\text{C.33})$$

Since  $m$  and  $b (= \bar{y})$  are independent, while  $m$  and  $c$  are not, the best line should be written as

$$y = (m \pm \Delta m)(x - \bar{x}) + b \pm \Delta b, \quad (\text{C.34})$$

and not as

$$y = (m \pm \Delta m)x + c \pm \Delta c. \quad (\text{C.35})$$

Equation (C.34) implies correctly that the error  $\Delta y$  in the best value of  $y$  at any value of  $x$  is given by

$$(\Delta y)^2 = (\Delta b)^2 + (x - \bar{x})^2 (\Delta m)^2. \quad (\text{C.36})$$

The best line may therefore be regarded as pivoting about the centre of gravity of the points – G in Fig. C.2. The errors in the  $y$  value of the pivot and in the slope of the line contribute independently to  $\Delta y$ . Equation (C.35) implies incorrectly that the pivot point is H.

## Appendix D

### The binomial and Poisson distributions

The Poisson distribution may be regarded as a limiting case of the binomial distribution, so we consider the latter first.

#### D.1 Binomial distribution

(a) *Derivation.* Suppose an event can have only two outcomes which we denote by A and B. Let the probability of outcome A be  $p$ . Then the probability of outcome B is  $q = 1 - p$ . If the event occurs  $N$  times, the probability  $w_N(n)$  that A comes up  $n$  times and B  $N - n$  times is equal to the number of ways of selecting  $n$  objects from  $N$ , i.e.  ${}_N C_n$ , times the probability that the first  $n$  events give A and the remaining  $N - n$  events give B. Therefore

$$w_N(n) = \frac{N!}{n!(N-n)!} p^n q^{N-n}. \quad (\text{D.1})$$

This probability function is known as the binomial distribution. It is specified by giving the values of the two parameters  $N$  and  $p$ . Figure D.1 shows the distribution for  $N = 10$ ,  $p = \frac{1}{3}$ .

If we sum  $w_N(n)$  from  $n = 0$  to  $n = N$ , the result must be unity. To check that this is so we use a mathematical trick, which will also be useful later. We define a function  $g(z)$  by

$$g(z) = (q + zp)^N. \quad (\text{D.2})$$

Then the coefficient of  $z^n$  is just  $w_N(n)$ , thus

$$\begin{aligned} g(z) &= (q + zp)^N = q^N + zNpq^{N-1} + z^2 \frac{N(N-1)}{2!} p^2 q^{N-2} + \dots + z^N p^N \\ &= w_N(0) + zw_N(1) + z^2 w_N(2) + \dots + z^N w_N(N). \end{aligned} \quad (\text{D.3})$$

$\sum_{n=0}^N w_N(n)$  is obtained from the last line by putting  $z = 1$ . Therefore

$$\sum_{n=0}^N w_N(n) = g(1) = (q + p)^N = 1, \quad \text{since } q + p = 1. \quad (\text{D.4})$$

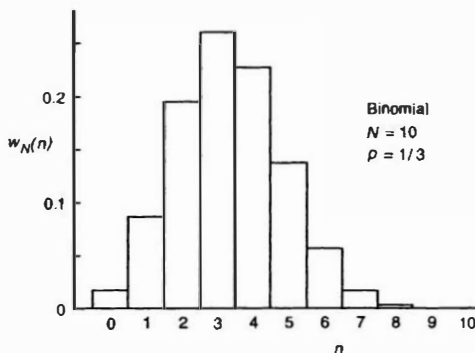


Fig. D.1. The binomial distribution for  $N = 10$ ,  $p = \frac{1}{3}$ .

(b) *Mean value of  $n$ .* The mean value of  $n$  for the distribution is

$$\begin{aligned}
 \langle n \rangle &= \sum_{n=0}^N n w_N(n) = \left( \frac{dg}{dz} \right)_{z=1} \quad (\text{from D.3}) \\
 &= Np(q+p)^{N-1} \\
 &= Np.
 \end{aligned} \tag{D.5}$$

We might have guessed this result. If the probability of A coming up in a single event is  $p$ , we would expect the average number of As in  $N$  events to be  $Np$ .

(c) *Standard deviation.* The standard deviation  $\sigma$  is given by

$$\begin{aligned}
 \sigma^2 &= \langle (n - \langle n \rangle)^2 \rangle = \langle n^2 - 2n\langle n \rangle + \langle n \rangle^2 \rangle \\
 &= \langle n(n-1) \rangle + \langle n \rangle - \langle n \rangle^2.
 \end{aligned} \tag{D.6}$$

Now

$$\begin{aligned}
 \langle n(n-1) \rangle &= \sum_{n=0}^N n(n-1) w_N(n) = \left( \frac{d^2g}{dz^2} \right)_{z=1} \\
 &= N(N-1)p^2(q+p)^{N-2} \\
 &= N(N-1)p^2.
 \end{aligned} \tag{D.7}$$

From (D.5), (D.6), and (D.7)

$$\begin{aligned}
 \sigma^2 &= N(N-1)p^2 + Np - N^2p^2 \\
 &= Np(1-p),
 \end{aligned} \tag{D.8}$$

or



$$\sigma = \sqrt{Npq}. \quad (\text{D.9})$$

As  $Np$  becomes large compared to unity, the standard deviation becomes small compared to the mean. The binomial distribution becomes more and more symmetric, and the points tend to lie on a Gaussian with mean value  $Np$  and standard deviation  $\sqrt{Npq}$ .

## D.2 Poisson distribution

(a) *Derivation.* The Poisson distribution is the limiting case of the binomial distribution when  $N$  tends to infinity and  $p$  tends to zero, in such a way that the product  $Np$  is equal to a finite constant which we denote by  $a$ . We require an expression for the probability  $w_a(n)$  that event A occurs  $n$  times.

$$\begin{aligned} w_a(n) &= \lim_{\substack{N \rightarrow \infty \\ p \rightarrow 0 \\ Np = a}} \frac{N!}{n!(N-n)!} p^n q^{N-n} \\ &= \frac{a^n}{n!} \lim_{N \rightarrow \infty} \frac{N(N-1) \cdots (N-n+1)}{N^n} q^{N-n}. \end{aligned} \quad (\text{D.10})$$

As  $N$  tends to infinity, the quantity

$$\frac{N(N-1) \cdots (N-n+1)}{N^n}$$

tends to unity. (Remember that  $n$  is finite, and therefore small compared with  $N$ .)

Also

$$\begin{aligned} q^{N-n} &= (1-p)^{N-n} = \left(1 - \frac{a}{N}\right)^{N-n} \\ &= \frac{\left(1 - \frac{a}{N}\right)^N}{\left(1 - \frac{a}{N}\right)^n}. \end{aligned} \quad (\text{D.11})$$

As  $N$  tends to infinity the numerator tends to  $\exp(-a)$  and the denominator to unity.

Collecting these results we have

$$w_a(n) = \exp(-a) \frac{a^n}{n!}, \quad (\text{D.12})$$

which is the Poisson distribution. Notice that, whereas the binomial distribution is specified by two parameters, the Poisson distribution is specified by the single parameter  $a$ .

We can check that summing  $w_a(n)$  over  $n$  gives unity.

$$\begin{aligned}\sum_{n=0}^{\infty} w_a(n) &= \exp(-a) \sum_{n=0}^{\infty} \frac{a^n}{n!} \\ &= \exp(-a) \exp a = 1.\end{aligned}\tag{D.13}$$

(b) *Mean value of  $n$  and standard deviation.* We may use results of the binomial distribution to obtain the mean value of  $n$  and the standard deviation.

$$\langle n \rangle = Np = a.\tag{D.14}$$

$$\sigma = \sqrt{Npq} = \sqrt{a},\tag{D.15}$$

since  $q = 1$  in the limit. As  $a$  becomes large, the Poisson distribution becomes more and more symmetric, and the points tend to lie on a Gaussian with mean  $a$  and standard deviation  $\sqrt{a}$ .

(c) *Application.* The Poisson distribution applies to the counting of particles when the average rate of arrival is constant – a common situation in atomic and nuclear physics. Suppose we are counting electrons with a scintillation counter, and we record the numbers that arrive in successive periods of, say, 10 seconds. The numbers form a Poisson distribution.

To see how the situation relates to our definition of the Poisson distribution, imagine that the 10-second interval is divided into  $N$  sub-intervals, where  $N$  is very large – say  $10^8$ . Suppose that  $a$ , the average number of electrons recorded in the 10-second intervals, is 5.3. Then the probability of an electron arriving in any particular sub-interval is

$$p = \frac{a}{N} = 5.3 \times 10^{-8}.\tag{D.16}$$

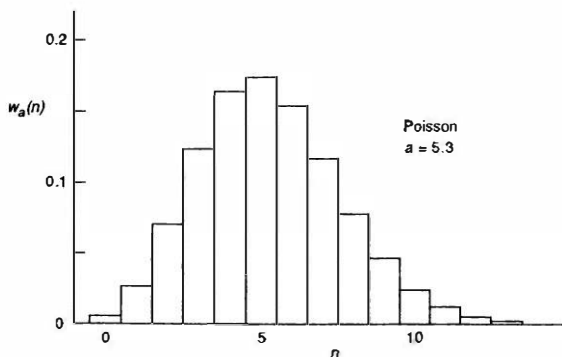


Fig. D.2. The Poisson distribution for  $a = 5.3$ .

Outcome A in this situation is the arrival of an electron in the sub-interval; outcome B is the non-arrival of an electron. ( $p$  is so small that we ignore the possibility of two electrons arriving in the same sub-interval.)

The probability of  $n$  electrons arriving in a 10-second interval is

$$w_{5.3}(n) = \exp(-5.3) \frac{(5.3)^n}{n!}. \quad (\text{D.17})$$

The standard deviation is  $\sqrt{5.3}$ . The Poisson distribution for  $a = 5.3$  is shown in Fig. D.2.

## Appendix E

### The $\chi^2$ distribution – test of goodness of fit

#### E.1 Introduction

Suppose we have a known function  $Y = Y(x)$  and a set of  $n$  pairs of measurements

$$(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n). \quad (\text{E.1})$$

The  $x$  values are known exactly, but the  $y$  values are subject to error. The question we are going to consider is how well the measured  $y$  values fit the function  $Y(x)$ .

Imagine that we have many sets of  $n$  pairs of measurements, the values  $x_1, x_2, \dots, x_n$  being the same for all the sets. If we look at the  $i$ th pair  $x_i, y_i$  for all the sets,  $x_i$  is the same throughout, but the value of  $y_i$  varies. The basic assumption in the calculation is that the  $y_i$  values form a Gaussian distribution. The mean of the distribution is  $Y_i = Y(x_i)$ , and the standard deviation is  $\sigma_i$  (assumed known). The situation is indicated in Fig. E.1. We start with a hypothesis, called the *null*

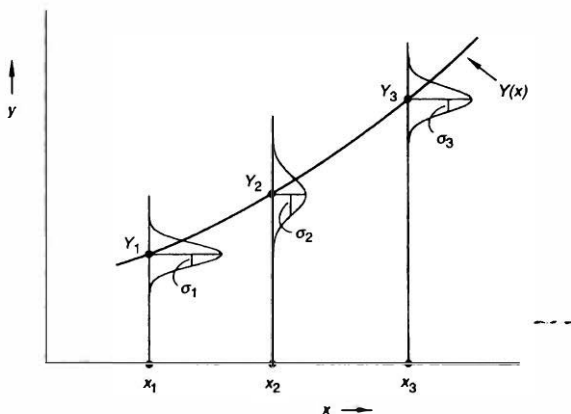


Fig. E.1. Repetition of measurements with a fixed set of  $x$  values for the function  $Y = Y(x)$ . It is assumed that the values at  $x_i$  have a Gaussian distribution centred on  $Y_i = Y(x_i)$  with standard deviation  $\sigma_i$ .

*hypothesis*, which is that the set of measurements in (E.1) is a random selection from the  $n$  Gaussian distributions.

In order to test the hypothesis we define a quantity  $\chi^2$  by

$$\chi^2 = \sum \left( \frac{y_i - Y_i}{\sigma_i} \right)^2. \quad (\text{E.2})$$

(We continue with the convention that, whenever the symbol  $\sum$  appears, the summation is to be taken from  $i = 1$  to  $i = n$ .) If the null hypothesis is correct, we expect  $(y_i - Y_i)/\sigma_i$  to be of the order of unity for each  $i$ . So  $\chi^2$  should be of the order of  $n$ . If  $\chi^2$  is very much larger than  $n$  there is a low probability that the null hypothesis is correct. We shall now put this statement on a quantitative basis, i.e. we shall calculate the probability that, if the null hypothesis is correct, we would obtain a value of  $\chi^2$  as large as, or larger, than the value given by (E.2). To do this we need the distribution function of  $\chi^2$  for a given value of  $n$ . This is the function  $F_n(\chi^2)$  such that  $F_n(\chi^2) d(\chi^2)$  is the probability that  $\chi^2$  has a value between  $\chi^2$  and  $\chi^2 + d(\chi^2)$ .

## E.2 Derivation of $\chi^2$ distribution

In order to derive an expression for  $F_n(\chi^2)$ , it is convenient to derive first a closely related function, namely  $f_n(\chi)$ , where  $f_n(\chi) d\chi$  is the probability that  $\chi$  has a value between  $\chi$  and  $\chi + d\chi$ . (The positive value of  $\chi$  is always taken.)

Suppose to start with we make only a single measurement  $x_1, y_1$ . We know from (3.34), p. 18, that for a Gaussian distribution the probability of  $y_1$  lying in the interval  $y_1$  to  $y_1 + dy_1$  is

$$f(y_1) dy_1 = \sqrt{\left(\frac{2}{\pi}\right)} \exp\left\{-(y_1 - Y_1)^2/2\sigma_1^2\right\} d\left(\frac{y_1}{\sigma_1}\right). \quad (\text{E.3})$$

The function  $f(y_1)$  has the same value for  $\pm(y_1 - Y_1)$ . We take only the positive value of  $(y_1 - Y_1)$ , but have included the negative value by doubling the expression in (3.34). The range of integration of  $f(y_1)$  is thus 0 to  $\infty$ .

Now suppose we make two measurements,  $x_1, y_1$  and  $x_2, y_2$ . Since the two measurements are independent of each other, the probability of obtaining  $y_1$  in the range  $y_1$  to  $y_1 + dy_1$ , and  $y_2$  in the range  $y_2$  to  $y_2 + dy_2$  is the product of the separate probability functions, and is thus

$$\begin{aligned} f(y_1, y_2) dy_1 dy_2 &= \frac{2}{\pi} \exp\left\{-(y_1 - Y_1)^2/2\sigma_1^2\right\} \\ &\quad \times \exp\left\{-(y_2 - Y_2)^2/2\sigma_2^2\right\} d\left(\frac{y_1}{\sigma_1}\right) d\left(\frac{y_2}{\sigma_2}\right) \\ &= \frac{2}{\pi} \exp(-\chi^2/2) d\left(\frac{y_1}{\sigma_1}\right) d\left(\frac{y_2}{\sigma_2}\right), \end{aligned} \quad (\text{E.4})$$

where

$$\chi^2 = \left(\frac{y_1 - Y_1}{\sigma_1}\right)^2 + \left(\frac{y_2 - Y_2}{\sigma_2}\right)^2. \quad (\text{E.5})$$

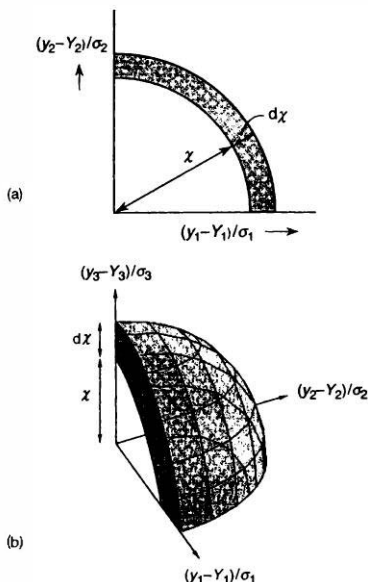


Fig. E.2. Diagrams showing regions of constant  $\chi$  for (a) two measurements and (b) three measurements.

However, we are not interested in the particular combination of  $y_1$  and  $y_2$  that gives the value of  $\chi$ , but in all  $y_1, y_2$  combinations that give the same value of  $\chi$ . Consider a two-dimensional plot of  $(y_1 - Y_1)/\sigma_1$  and  $(y_2 - Y_2)/\sigma_2$  - Fig. E.2a.  $\chi$  is constant over a quadrant of a circle of radius  $\chi$ . (We take only a quadrant because, as before, we take the positive values of  $(y_1 - Y_1)/\sigma_1$  and  $(y_2 - Y_2)/\sigma_2$ .) Therefore, the probability that  $\chi$  has the value  $\chi$  to  $\chi + d\chi$  is obtained by replacing the element  $d(y_1/\sigma_1) d(y_2/\sigma_2)$  in (E.4) by  $(2\pi\chi/4)d\chi$ , the area between two quadrants of radii  $\chi$  and  $\chi + d\chi$ . Thus

$$f_2(\chi) d\chi = \frac{\pi}{2} \chi \frac{2}{\pi} \exp(-\chi^2/2) d\chi = \chi \exp(-\chi^2/2) d\chi. \quad (\text{E.6})$$

A similar calculation applies for three measurements.  $\chi^2$  is the sum of three terms according to (E.2). When we multiply the three Gaussian probability functions we obtain a factor proportional to  $\exp(-\chi^2/2)$ , and this is multiplied by the volume of the shell between octants of radii  $\chi$  and  $\chi + d\chi$ , which is  $4\pi\chi^2 d\chi/8$  - Fig. E.2b. The result is

$$f_3(\chi) = \sqrt{\left(\frac{2}{\pi}\right)} \chi^2 \exp(-\chi^2/2). \quad (\text{E.7})$$

We are now in a position to give the expression for  $f(\chi)$  for general values of  $n$ . The method we have used so far can be applied for all values of  $n$ . The expression for  $f_n(\chi)$  is obtained by multiplying the factor  $\exp(-\chi^2/2)$  by the 'volume' of a 'shell' in  $n$ -dimensional  $\chi$ -space, which is proportional to  $\chi^{n-1} d\chi$ . Thus, for general values of  $n$  the probability that  $\chi$  lies between  $\chi$  and  $\chi + d\chi$  is

$$f_n(\chi) d\chi = C_n \chi^{n-1} \exp(-\chi^2/2) d\chi, \quad (\text{E.8})$$

where  $C_n$  is a constant that ensures that

$$C_n \int_0^\infty \chi^{n-1} \exp(-\chi^2/2) d\chi = 1. \quad (\text{E.9})$$

The values of the constants  $C_n$  are readily obtained. We have already seen that

$$C_1 = \sqrt{\left(\frac{2}{\pi}\right)}, \quad C_2 = 1, \quad C_3 = \sqrt{\left(\frac{2}{\pi}\right)}. \quad (\text{E.10})$$

For  $n > 2$ , the expressions for  $C_n$  are obtained from the expressions for the integral  $I_n$  given in (A.11) and (A.12), p. 163, with  $\sigma = 1$ . Since

$$C_n = \frac{1}{I_{n-1}}, \quad (\text{E.11})$$

we have

$$\begin{aligned} C_n &= \sqrt{\left(\frac{2}{\pi}\right)} / \{1 \cdot 3 \cdot 5 \cdot \dots \cdot (n-2)\} \quad n \text{ odd, and } \geq 3, \\ C_n &= 1 / \{2 \cdot 4 \cdot 6 \cdot \dots \cdot (n-2)\} \quad n \text{ even, and } \geq 4. \end{aligned} \quad (\text{E.12})$$

These equations show that for all values of  $n$

$$C_{n+2} = \frac{C_n}{n}. \quad (\text{E.13})$$

For a given value of  $n$ , the mean value of  $\chi^2$  is equal to  $n$ . This is readily shown as follows.

$$\begin{aligned} \langle \chi^2 \rangle &= C_n \int_0^\infty \chi^{n+1} \exp(-\chi^2/2) d\chi \\ &= \frac{C_n}{C_{n+2}} C_{n+2} \int_0^\infty \chi^{n+1} \exp(-\chi^2/2) d\chi = \frac{C_n}{C_{n+2}} = n. \end{aligned} \quad (\text{E.14})$$

In the last line we have made use of (E.9) and (E.13). This result is to be expected. By definition the mean value of  $(y_i - Y_i)^2$  is  $\sigma_i^2$ . Thus the mean value of each term in the sum in (E.2) is unity, and there are  $n$  terms in the sum.

Finally, we deduce the function  $F_n(\chi^2)$  from the expression for  $f_n(\chi)$  in (E.8). We have

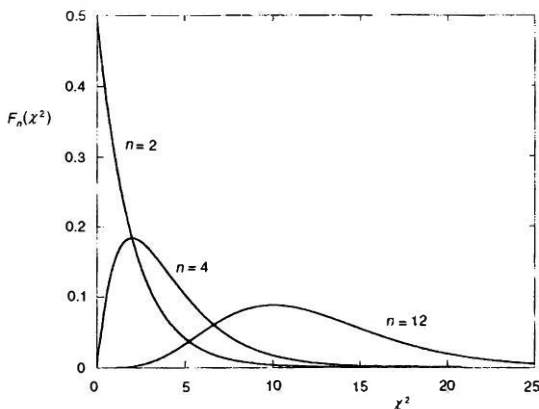


Fig. E.3. The distribution function  $F_n(\chi^2)$  for  $n=2, 4, 12$ .

$$F_n(\chi^2) d(\chi^2) = f_n(\chi) d\chi, \quad \text{and} \quad d(\chi^2) = 2\chi d\chi. \quad (\text{E.15})$$

$$\text{So} \quad F_n(\chi^2) = \frac{1}{2\chi} f_n(\chi) = \frac{1}{2} C_n (\chi^2)^{n/2-1} \exp(-\chi^2/2). \quad (\text{E.16})$$

The function  $F_n(\chi^2)$  is shown in Fig. E.3 for  $n=2, 4$ , and  $12$ .

### E.3 The function $P_n(\chi^2)$

Suppose that we have made a set of measurements and obtained a value for  $\chi^2$ . The probability that  $\chi^2$  should be as large or larger than the measured value is

$$P_n(\chi^2) = \int_{\chi^2}^{\infty} F_n(t) dt = \frac{1}{2} C_n \int_{\chi^2}^{\infty} t^{n/2-1} \exp(-t/2) dt. \quad (\text{E.17})$$

It is equal to the shaded area in Fig. E.4 divided by the total area under the distribution function  $F_n(t)$ .

For even values of  $n$ , the integration in (E.17) can be done explicitly and leads to the result

$$P_n(u) = \left( 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots + \frac{u^m}{m!} \right) \exp(-u), \quad (\text{E.18})$$

where  $u = \frac{\chi^2}{2}$ , and  $m = \frac{n}{2} - 1$ .

For odd values of  $n$ ,  $P_n(\chi)$  must be calculated by numerical methods. The function  $P_n(\chi^2)$  is shown in Fig. E.5 for  $n=2, 4$ , and  $12$ .



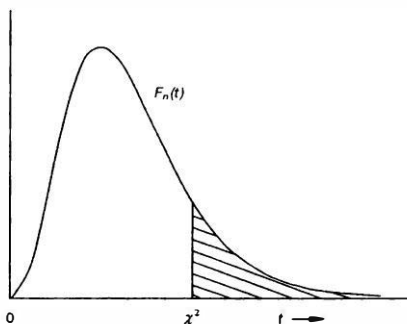


Fig. E.4.  $P_n(\chi^2)$ , the probability of obtaining a value greater than or equal to the observed value  $\chi^2$ , is the ratio of the shaded area to the total area under the distribution function  $F_n(t)$ . (The function  $F_n(t)$  in the figure corresponds to  $n = 8$ .)

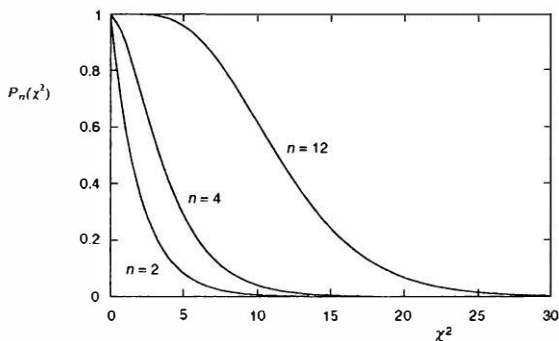


Fig. E.5. The probability function  $P_n(\chi^2)$  for  $n = 2, 4, 12$ .

#### E.4 Degrees of freedom

So far we have assumed that the parameters that specify the function  $Y(x)$  are independent of the measurements  $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ . However, it is usually the case that some or all of the parameters are calculated from the measurements themselves. The deviations  $y_1 - Y_1, y_2 - Y_2$ , etc. are then not independent. There are relations between them which tend to reduce their values.

We have already had examples of this. In (3.14), p. 15, the quantity  $s^2$  is obtained from the sum of the squares of the deviations of a set of  $n$  readings taken

from the mean of the set. But the  $n$  deviations are not independent; they satisfy the relation  $\sum d_i = 0$ . If we are told the values of all but one of the deviations, we can calculate the other one from this relation, i.e. we have, not  $n$ , but  $n-1$  independent deviations. For this reason the quantity  $s^2$  in (3.14) is on average less than the variance  $\sigma^2$ , as we saw in section 3.4 (d). To obtain an unbiased estimate of  $\sigma^2$ , we divide  $\sum d_i^2$  by  $n-1$ , and not by  $n$ .

The same argument applies to fitting the best straight line  $y = mx + c$  to a set of  $n$  pairs of  $x_i, y_i$  values. We calculate the sum of the squares of the deviations from the best straight line  $y = mx + c$ , but we have used the measurements themselves to determine the parameters  $m$  and  $c$ . The deviations therefore satisfy the two relations

$$\sum d_i = 0, \quad \text{and} \quad \sum x_i d_i = 0 \quad (\text{E.19})$$

– see exercise 4.6. This reduces the number of independent deviations from  $n$  to  $n-2$ , which is reflected in the result that, to obtain an unbiased estimate of  $\sigma^2$ , the variance of the measurements about the true straight line, we divide the sum of the squares of the deviations from the best line  $y = mx + c$  by  $n-2$ ; see (C.30), p. 170.

In general, if there are  $n$  pairs of measurements and  $r$  parameters in the function  $Y(x)$  that have been determined by the measurements, the number of *independent* deviations is

$$v = n - r. \quad (\text{E.20})$$

The quantity  $v$  is called the number of *degrees of freedom*, and  $r$  is termed the number of *constraints*. The effect of the constraints in the probability interpretation of  $\chi^2$  is that the index  $n$  in (E.16) is replaced by  $v$ , i.e.  $\chi^2$  is calculated from (E.2) for all the  $n$  measurements, and the probability that  $\chi^2$  has the value between  $\chi^2$  and  $\chi^2 + d(\chi^2)$  is

$$F_r(\chi^2) d(\chi^2) = \frac{1}{2} C_r(\chi^2)^{v/2-1} \exp(-\chi^2/2) d(\chi^2). \quad (\text{E.21})$$

A formal proof of this result is given in several textbooks of statistics – see for example Weatherburn 1961, p. 166.

The values of  $\chi^2$  for values of  $v$  and  $P$  are given in Table H.2.

### E.5 Test of goodness of fit

We now consider another, and important, situation in which the  $\chi^2$  distribution applies. It has the simplifying feature that the standard deviations  $\sigma_i$  do not have to be known separately from the  $Y_i$  values, but are obtained from them.

Suppose we have a set of  $N$  observations each of which falls into one of  $n$  classes. The number of observations that fall into the  $i$ th class is  $y_i$ , and as before, we want to calculate the probability that the set of  $y_i$  values comes from a set of

Gaussian distributions centred on a set of  $Y_i$  values. In the previous situation, the  $Y_i$  values came from the function  $Y(x)$ , and the value of  $\chi^2$  is a measure of how well the measurements fit the function. In the present case, instead of the function  $Y(x)$ , we have a given probability  $p_i$  that an observation falls into the  $i$ th class.  $Y_i$  is related to  $p_i$  as we show below (E.22), and the value of  $\chi^2$  is a measure of how well the values of  $y_i$  fit the set of  $p_i$  values. Note that in the previous situation  $y_i$  and  $Y_i$  are physical quantities and, in general, have dimensions;  $\sigma_i$  has the same dimensions, so  $\chi^2$  is dimensionless. In the present case  $y_i$ ,  $Y_i$  and  $\sigma_i$  are dimensionless numbers,  $y_i$  being an integer.

Suppose we make many sets of  $N$  observations and look at the distribution of the  $y_i$  values for the  $i$ th class. This will be a binomial, given by (D.1), p. 171. In that equation, outcome A, with probability  $p_i$ , is obtaining an observation in the  $i$ th class; outcome B, with probability  $q_i = 1 - p_i$ , is obtaining an observation in any one of the other classes. From (D.5) and (D.8) the distribution of  $y_i$  values has

$$\text{mean} \quad Y_i = Np_i \quad (\text{E.22})$$

$$\text{variance} \quad \sigma_i^2 = Np_i q_i. \quad (\text{E.23})$$

Provided  $Np_i$  is fairly large compared to unity, the standard deviation is small compared to the mean and, as stated in Appendix D, the binomial distribution then approximates to a Gaussian, which is consistent with the basic assumption in section E.1.

To obtain an expression for  $\chi^2$  that satisfies the distribution given in (E.21) it might be thought that, from (E.2) and (E.23), we should take

$$\chi^2 = \sum \frac{(y_i - Y_i)^2}{Np_i q_i}. \quad (\text{E.24})$$

However, that is not correct, because it ignores the fact that, for each set of  $N$  observations, we have the constraint

$$\sum y_i = N, \quad (\text{E.25})$$

which tends to reduce the sum of the squares of the deviations. The correct expression for  $\chi^2$ , i.e. the one that satisfies the distribution given in (E.21), is

$$\chi^2 = \sum \frac{(y_i - Y_i)^2}{Np_i}. \quad (\text{E.26})$$

A formal proof of this is given in Stuart and Ord 1994, p. 520. We give here a simple argument to show the plausibility of the result.

From the definition of  $p_i$  we have the relation

$$\sum p_i = 1. \quad (\text{E.27})$$

Assume all the  $p_i$  and hence all the  $q_i$ , are equal, i.e.

$$p_i = p = \frac{1}{n}, \quad \text{and} \quad q_i = q = 1 - p \quad (\text{E.28})$$

for all  $i$ . From the discussion in section E.4, the effect of the constraint in (E.25) is that the expression for  $\chi^2$  in (E.24) should be multiplied by the factor

$$\frac{n-1}{n} = 1 - \frac{1}{n} = 1 - p = q, \quad (\text{E.29})$$

which gives (E.26).

It should be noted that the constraint reduces the number of degrees of freedom to  $n-1$ . The number is still further reduced if some of the parameters that determine the probabilities  $p_i$  come from the data themselves.

From (E.22) and (E.26)

$$\chi^2 = \sum \frac{(y_i - Y_i)^2}{Y_i}. \quad (\text{E.30})$$

A common notation is to denote the quantity  $Y_i$  by  $E_i$  (the expected value) and  $y_i$  by  $O_i$  (the observed value). Thus (E.30) is written as

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i}. \quad (\text{E.31})$$

Note that the interpretation of  $\chi^2$  depends on a Gaussian distribution for  $y_i$ , and this is valid only if  $Y_i$  is not too small compared to unity. If this condition is not satisfied – a minimum value of five is conventionally taken – the values for two or more intervals should be taken together.

The result in (E.30) was first derived by Pearson (1900) and is known as the  $\chi^2$  test of goodness of fit.

## E.6 Worked examples

(a) **Neutron lifetime.** Using the expression in (E.2) we calculate the value of  $\chi^2$  for the four values of the neutron lifetime  $\tau$  given in exercise 4.5 (pp. 42 and 198) to see if they are consistent with each other, taking into account their standard errors. The null hypothesis is that each of the four values of  $\tau$  is a random sample from a Gaussian distribution, the mean values of the four distributions being equal to the weighted mean  $\bar{\tau}$  of the four values of  $\tau$ . The standard deviation of each distribution is equal to the standard error  $\Delta\tau_i$  in each value of  $\tau$ . In the present case the variable  $x$  is simply the index  $i$ . Thus

$$\chi^2 = \sum \left( \frac{\tau_i - \bar{\tau}}{\Delta\tau_i} \right)^2. \quad (\text{E.32})$$

The calculation is set out in Table E.I. The value of  $\chi^2$  is 4.30. There are  $n=4$  terms in the sum. The value of  $\bar{\tau}$  comes from the experimental values, so the

Table E.1.  $\chi^2$  analysis of measurements of neutron lifetime  $\tau$ .  
Weighted mean  $\bar{\tau} = 886.5$  s.

$i$	$\tau/\text{s}$	$\Delta\tau/\text{s}$	$\left(\frac{\tau - \bar{\tau}}{\Delta\tau}\right)^2$
1	887.6	3.0	0.13
2	893.5	5.3	1.74
3	888.4	3.3	0.33
4	882.6	2.7	2.09
sum			4.30

number of constraints is  $r = 1$ . The number of degrees of freedom is therefore  $\nu = n - r = 3$ . From Table H.2 we see that for  $\chi^2 = 4.30$ , and  $\nu = 3$ , the value of  $P$  is just less than 0.25. Since this is not very small compared to unity, we conclude that the four values of  $\tau$  are reasonably consistent. Had  $P$  been small, this would have indicated that there was probably a systematic error in one or more of the values of  $\tau$ , in which case the weighted mean would not have been a significant quantity.

(b) **Radioactive background.** As an example of the goodness of fit test we calculate the value of  $\chi^2$  for a set of measurements of the background counts in a radioactivity experiment. The radioactive source was removed for this part of the experiment, and the counts were due to cosmic rays and other forms of stray radiation. The number of counts  $k$  in a period of 10 seconds was recorded, and this measurement was made 400 times. The calculation of  $\chi^2$  is readily done with a spreadsheet and is shown in Table E.2. A histogram of the measurements is inserted in the first two columns.  $O$  is the number of times that  $k$  counts were recorded. If the counts occur at random times we expect the histogram to follow a Poisson distribution – see p. 173. We therefore analyse the results to see how well they fit this distribution. The mean number of counts in a 10-second period is  $a = 6.38$ . From (D.12), p. 173, the expected number of times that  $k$  counts are observed is

$$E(k) = N \exp(-a) \frac{a^k}{k!}, \quad (\text{E.33})$$

where  $N = 400$ . The calculated values of  $E(k)$  are given in the third column. In order to obtain values of  $E$  greater than 5, the  $O$  and  $E$  values for  $k = 0$  and 1 have been combined; the values for  $k$  from 13 to 16 have been similarly combined.

The values of  $(O - E)^2/E$  in the fourth column sum to 8.18. There are  $n = 13$  terms in the sum. The values of  $N$  and  $a$  have gone into the calculation of  $E(k)$ , so there are  $r = 2$  constraints. Thus the number of degrees of freedom is  $\nu = n - r = 11$ . Table H.2 shows that, for  $\chi^2 = 8.18$  and  $\nu = 11$ , the value of  $P$  is

Table E.2.  $\chi^2$  analysis of measurements of number of counts in a 10-second period

$k$	$O$		$E$		$(O - E)^2/E$
0	0	2	0.7	5.0	1.80
1	2		4.3		
2	16		13.8		0.35
3	30		29.3		0.01
4	44		46.8		0.17
5	62		59.7		0.09
6	71		63.5		0.88
7	61		57.9		0.17
8	42		46.2		0.38
9	28		32.7		0.68
10	24		20.9		0.47
11	13		12.1		0.07
12	2		6.4		3.06
13	3	5	3.2	5.5	0.05
14	2		1.4		
15	0		0.6		
16	0		0.2		
sum	400		399.9		8.18

about 0.7, so the Poisson distribution is a good fit to the data, i.e. the latter are consistent with the hypothesis that the counts occur at random times. In fact we do not have to refer to Table H.2 to reach this conclusion. If the value of  $\chi^2$  is not very different from that of  $\nu$ , the data are consistent with the null hypothesis.

## E.7 Comments

(1) As we have seen, a value of  $P_\nu(\chi^2)$  small compared to unity is evidence against the null hypothesis. A common (but arbitrary) value of  $P_\nu(\chi^2) = 0.05$  is taken for this purpose, and the discrepancy with the null hypothesis is said to be significant at the 5% level.

(2) A very high value of  $\chi^2$  indicates that the null hypothesis is probably not correct. However, a very low value of  $\chi^2$ , giving a value of  $P_\nu(\chi^2)$  very close to unity, does *not* mean that the null hypothesis is probably correct. It has a quite different significance. The quantity  $1 - P_\nu(\chi^2)$  is the probability that the value of  $\chi^2$  is as low or lower than the calculated value. So a value of  $P_\nu(\chi^2)$  close to unity means that the experimental data are an improbably good fit to the function  $Y(x)$ , or the probabilities  $p_i$ . It indicates that there is probably something wrong with the data or with the calculation of  $\chi^2$ .

In the former category is the possibility that the experimenter has *selected* results that agree with the null hypothesis. The calculation of  $\chi^2$  may involve more innocent mistakes. If (E.2) is used for the purpose, it may be that the

assumed values of  $\sigma_i^2$  are too large. Another possibility is that the data appear to fit the function  $Y(x)$  very well because a number of parameters in the function have been obtained from the data, and allowance for the constraints has not been made, so the assumed number of degrees of freedom is too high.

If all these possibilities have been eliminated and the value of  $\chi^2$  is still improbably low, then, assuming the data are from your own measurements, the standard remedy should be applied, namely make more measurements.

## Appendix F

### SI units

The system of units used throughout this book is known as SI, an abbreviation for *Système International d'Unités*. It is a comprehensive, logical system, designed for use in all branches of science and technology. It was formally approved in 1960 by the General Conference of Weights and Measures, the international organization responsible for maintaining standards of measurement. Apart from its intrinsic merits, it has the great advantage that *one* system covers all situations – theoretical and practical.

A full account of SI will be found in a publication of the National Physical Laboratory (Bell 1993). The following are the essential features of the system.

(1) SI is a metric system. There are seven base units (see next section), the metre and kilogram replacing the centimetre and gram of the old c.g.s. system.

(2) The derived units are directly related to the base units. For example, the unit of acceleration is  $1 \text{ m s}^{-2}$ . The unit of force is the newton, which is the force required to give a body of mass  $1 \text{ kg}$  an acceleration of  $1 \text{ m s}^{-2}$ . The unit of energy is the joule, which is the work done when a force of  $1 \text{ N}$  moves a body a distance of  $1 \text{ m}$ .

The use of auxiliary units is discouraged in SI. Thus the unit of pressure, the pascal, is  $1 \text{ N m}^{-2}$ ; the atmosphere and the torr are not used. Similarly the calorie is not used; all forms of energy are measured in joules. (However, the electron volt is a permitted unit.)

(3) Electrical units are rationalized and are obtained by assigning the value  $4\pi \times 10^{-7} \text{ H m}^{-1}$  to  $\mu_0$ , the permeability of a vacuum. This leads to the ampere – one of the seven base units – as the unit of current. The other electrical units are derived directly from the base units and are identical with the practical units. (See Duffin 1990 for a clear discussion of electrical units.)



## SI units – names and symbols

Quantity	Unit	Symbol	Relation to other units
<b>Base units</b>			
length	metre	m	
mass	kilogram	kg	
time	second	s	
electric current	ampere	A	
thermodynamic temperature	kelvin	K	
amount of substance	mole	mol	
luminous intensity	candela	cd	
<b>Derived units with special names</b>			
force	newton	N	$\text{kg m s}^{-2}$
pressure	pascal	Pa	$\text{N m}^{-2}$
energy	joule	J	$\text{N m}$
power	watt	W	$\text{J s}^{-1}$
electric charge	coulomb	C	$\text{A s}$
potential	volt	V	$\text{J C}^{-1}$
resistance	ohm	$\Omega$	$\text{V A}^{-1}$
capacitance	farad	F	$\text{C V}^{-1} = \text{s } \Omega^{-1}$
magnetic flux	weber	Wb	$\text{V s}$
flux density	tesla	T	$\text{Wb m}^{-2}$
inductance	henry	H	$\text{Wb A}^{-1} = \Omega \text{ s}$
frequency	hertz	Hz	$\text{s}^{-1}$
temperature	degree Celsius	$^{\circ}\text{C}$	$t/^{\circ}\text{C} = T/\text{K} - 273.15$
luminous flux	lumen	lm	$\text{cd sr}$
illuminance	lux	lx	$\text{lm m}^{-2}$
activity (radioactive)	becquerel	Bq	$\text{s}^{-1}$
absorbed dose (of ionizing radiation)	gray	Gy	$\text{J kg}^{-1}$
plane angle	radian	rad	
solid angle	steradian	sr	

## Decimal factors

Factor	Prefix	Symbol	Factor	Prefix	Symbol
$10^{-1}$	deci	d	$10^1$	deca	da
$10^{-2}$	centi	c	$10^2$	hecto	h
$10^{-3}$	milli	m	$10^3$	kilo	k
$10^{-6}$	micro	$\mu$	$10^6$	mega	M
$10^{-9}$	nano	n	$10^9$	giga	G
$10^{-12}$	pico	p	$10^{12}$	tera	T
$10^{-15}$	femto	f	$10^{15}$	peta	P
$10^{-18}$	atto	a	$10^{18}$	exa	E
$10^{-21}$	zepto	z	$10^{21}$	zetta	Z
$10^{-24}$	yocto	y	$10^{24}$	yotta	Y

*Note.* The prefix is to be taken together with the unit *before* the power symbol operates, e.g.

$$1 \mu\text{m}^2 = 1(\mu\text{m})^2 = 10^{-12} \text{m}^2.$$

## Relation to c.g.s. units

*Electrical units*

$$1 \text{ A} = 10^{-1} \text{ e.m.u.} = c/10 \text{ e.s.u.}$$

$$1 \text{ V} = 10^8 \text{ e.m.u.} = 10^8/c \text{ e.s.u.}$$

$$1 \Omega = 10^9 \text{ e.m.u.} = 10^9/c^2 \text{ e.s.u.}$$

$$1 \text{ F} = 10^{-9} \text{ e.m.u.} = 10^{-9} c^2 \text{ e.s.u.}$$

$$1 \text{ H} = 10^9 \text{ e.m.u.} = 10^9/c^2 \text{ e.s.u.}$$

$$1 \text{ T} = 10^4 \text{ e.m.u. (gauss)}$$

$$1 \text{ Wb} = 10^8 \text{ e.m.u. (maxwell)}$$

$$1 \text{ A m}^{-1} = 4\pi \times 10^{-3} \text{ e.m.u. (oersted)}$$

$$(c \approx 3 \times 10^{10} \text{ in the above equations.})$$

*Other units*

length	1 parsec = $3.086 \times 10^{16} \text{ m}$
	1 light-year = $9.46 \times 10^{15} \text{ m}$
	1 micron = $10^{-6} \text{ m}$
	1 ångström = $10^{-10} \text{ m}$
	1 fermi = $10^{-15} \text{ m}$
area	1 barn = $10^{-28} \text{ m}^2$
volume	1 litre = $10^{-3} \text{ m}^3$
force	1 dyne = $10^{-5} \text{ N}$
energy	1 erg = $10^{-7} \text{ J}$
	1 calorie (IT) = $4.1868 \text{ J}$
	1 electron volt = $1.6022 \times 10^{-19} \text{ J}$
pressure	1 bar = $10^5 \text{ Pa}$
	1 atmosphere = $1.01325 \times 10^5 \text{ Pa}$
	1 torr = 1 mm of Hg = $133.322 \text{ Pa}$
viscosity	
	dynamic 1 poise = $10^{-1} \text{ Pa s}$
kinematic	1 stokes = $10^{-4} \text{ m}^2 \text{ s}^{-1}$

### Definitions of the SI base units

#### *Metre*

The *metre* is the length of the path travelled by light in vacuum during a time interval of  $1/299\,792\,458$  of a second.

#### *Kilogram*

The *kilogram* is the unit of mass; it is equal to the mass of the international prototype of the kilogram.

#### *Second*

The *second* is the duration of  $9\,192\,631\,770$  periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium-133 atom.

#### *Ampere*

The *ampere* is that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section, and placed 1 metre apart in vacuum, would produce between these conductors a force equal to  $2 \times 10^{-7}$  newton per metre of length.

#### *Kelvin*

The *kelvin*, unit of thermodynamic temperature, is the fraction  $1/273.16$  of the thermodynamic temperature of the triple point of water.

#### *Mole*

The *mole* is the amount of substance of a system which contains as many elementary entities as there are atoms in 0.012 kilogram of carbon-12. When the mole is used, the elementary entities must be specified and may be atoms, molecules, ions, electrons, other particles, or specified groups of such particles.

#### *Candela*

The *candela* is the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency  $540 \times 10^{12}$  hertz and that has a radiant intensity in that direction of  $1/683$  watt per steradian.

The above definition of the metre came into effect in 1983 and replaced the definition which assigned a certain value to the wavelength of one of the lines in the spectrum of the krypton-86 atom. Thus, the speed of light is now *defined* to be exactly  $299\,792\,458 \text{ m s}^{-1}$ .

# Appendix G

## Values of physical constants

Physical constant	Value	Fractional uncertainty
atomic mass unit	$m_u = 10^{-3}/N_A = 1.6605 \times 10^{-27} \text{ kg}$	$8 \times 10^{-8}$
energy equivalent	$m_u c^2 = 931.49 \text{ MeV}$	$4 \times 10^{-8}$
Avogadro constant	$N_A = 6.0221 \times 10^{23} \text{ mol}^{-1}$	$8 \times 10^{-8}$
Bohr magneton	$\mu_B = 9.2740 \times 10^{-24} \text{ J T}^{-1}$	$4 \times 10^{-8}$
Boltzmann constant	$k = 1.3807 \times 10^{-23} \text{ J K}^{-1}$	$2 \times 10^{-6}$
elementary charge	$e = 1.6022 \times 10^{-19} \text{ C}$	$4 \times 10^{-8}$
Faraday constant	$F = N_A e = 9.6485 \times 10^4 \text{ C mol}^{-1}$	$4 \times 10^{-8}$
fine-structure constant	$\alpha = 7.2974 \times 10^{-3}$	$4 \times 10^{-9}$
	$1/\alpha = 137.036$	$4 \times 10^{-9}$
gravitational constant	$G = 6.673 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$	$1.5 \times 10^{-3}$
gyromagnetic ratio of proton	$\gamma_p = 2.6752 \times 10^8 \text{ s}^{-1} \text{ T}^{-1}$	$4 \times 10^{-8}$
magnetic flux quantum	$\Phi_0 = h/2e = 2.0678 \times 10^{-15} \text{ Wb}$	$4 \times 10^{-8}$
mass of electron	$m_e = 9.1094 \times 10^{-31} \text{ kg}$	$8 \times 10^{-8}$
mass of neutron	$m_n = 1.6749 \times 10^{-27} \text{ kg}$	$8 \times 10^{-8}$
mass of proton	$m_p = 1.6726 \times 10^{-27} \text{ kg}$	$8 \times 10^{-8}$
molar gas constant	$R = N_A k = 8.3145 \text{ J K}^{-1} \text{ mol}^{-1}$	$2 \times 10^{-6}$
molar volume of ideal gas	$V_m = 22.414 \times 10^{-3} \text{ m}^3 \text{ mol}^{-1}$	$2 \times 10^{-6}$
nuclear magneton	$\mu_N = 5.0508 \times 10^{-27} \text{ J T}^{-1}$	$4 \times 10^{-8}$
permittivity of vacuum	$\epsilon_0 = 1/\mu_0 c^2 = 8.8542 \times 10^{-12} \text{ F m}^{-1}$	zero
Planck constant	$h = 6.6261 \times 10^{-34} \text{ J s}$	$8 \times 10^{-8}$
	$\hbar = h/2\pi = 1.0546 \times 10^{-34} \text{ J s}$	$8 \times 10^{-8}$
Rydberg constant	$R_\infty = 1.0974 \times 10^7 \text{ m}^{-1}$	$8 \times 10^{-12}$
speed of light	$c = 2.9979 \times 10^8 \text{ m s}^{-1}$	zero
Stefan-Boltzmann constant	$\sigma = 5.6704 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$	$7 \times 10^{-6}$
acceleration due to gravity	$g = (9.7803 + 0.0519 \sin^2 \phi - 3.1 \times 10^{-6} H) \text{ m s}^{-2}$	
	$\phi$ = latitude; $H$ = height above sea level in metres	

The values of the physical constants, other than  $g$ , are taken from Mohr and Taylor 2000. The expression for  $g$  is taken from Kaye and Laby 1995, p. 193, and gives values correct to about  $5 \times 10^{-4} \text{ m s}^{-2}$ .

# Appendix H

## Mathematical tables

**Table H.1. Values of the Gaussian function and the Gaussian integral function**

$$f(z) = \frac{1}{\sqrt{2\pi}} \exp(-z^2/2) \quad \phi(z) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^z \exp(-t^2/2) dt$$

$z$	$f(z)$	$\phi(z)$	$z$	$f(z)$	$\phi(z)$
0.0	0.3989	0.0000	2.0	0.0540	0.9545
0.1	0.3970	0.0797	2.1	0.0440	0.9643
0.2	0.3910	0.1585	2.2	0.0355	0.9722
0.3	0.3814	0.2358	2.3	0.0283	0.9786
0.4	0.3683	0.3108	2.4	0.0224	0.9836
0.5	0.3521	0.3829	2.5	0.0175	0.9876
0.6	0.3332	0.4515	2.6	0.0136	0.9907
0.7	0.3123	0.5161	2.7	0.0104	0.9931
0.8	0.2897	0.5763	2.8	0.0079	0.9949
0.9	0.2661	0.6319	2.9	0.0060	0.9963
1.0	0.2420	0.6827	3.0	0.0044	0.99730
1.1	0.2179	0.7287	3.1	0.0033	0.99806
1.2	0.1942	0.7699	3.2	0.0024	0.99863
1.3	0.1714	0.8064	3.3	0.0017	0.99903
1.4	0.1497	0.8385	3.4	0.0012	0.99933
1.5	0.1295	0.8664	3.5	0.0009	0.99953
1.6	0.1109	0.8904	3.6	0.0006	0.99968
1.7	0.0940	0.9109	3.7	0.0004	0.99978
1.8	0.0790	0.9281	3.8	0.0003	0.99986
1.9	0.0656	0.9426	3.9	0.0002	0.99990
			4.0	0.0001	0.99994

**Table H.2. Values of  $\chi^2$  for given  $\nu$  and  $P$** 

$P$  is the probability that, for  $\nu$  degrees of freedom,  $\chi^2$  should be as large or larger than the tabulated value.

$\nu$	$P$					
	0.99	0.975	0.95	0.90	0.75	0.50
1	0.00016	0.00098	0.0039	0.0158	0.102	0.455
2	0.0201	0.0506	0.103	0.211	0.575	1.386
3	0.115	0.216	0.352	0.584	1.213	2.366
4	0.297	0.484	0.711	1.064	1.923	3.357
5	0.554	0.831	1.145	1.610	2.675	4.351
6	0.87	1.24	1.64	2.20	3.45	5.35
7	1.24	1.69	2.17	2.83	4.25	6.35
8	1.65	2.18	2.73	3.49	5.07	7.34
9	2.09	2.70	3.33	4.17	5.90	8.34
10	2.56	3.25	3.94	4.87	6.74	9.34
11	3.05	3.82	4.57	5.58	7.58	10.34
12	3.57	4.40	5.23	6.30	8.44	11.34
13	4.11	5.01	5.89	7.04	9.30	12.34
14	4.66	5.63	6.57	7.79	10.17	13.34
15	5.23	6.26	7.26	8.55	11.04	14.34
16	5.81	6.91	7.96	9.31	11.91	15.34
17	6.41	7.56	8.67	10.09	12.79	16.34
18	7.01	8.23	9.39	10.86	13.68	17.34
19	7.63	8.91	10.12	11.65	14.56	18.34
20	8.26	9.59	10.85	12.44	15.45	19.34
22	9.54	10.98	12.34	14.04	17.24	21.34
24	10.86	12.40	13.85	15.66	19.04	23.34
26	12.20	13.84	15.38	17.29	20.84	25.34
28	13.56	15.31	16.93	18.94	22.66	27.34
30	14.95	16.79	18.49	20.60	23.57	29.34

<i>v</i>	<i>P</i>				
	0.25	0.10	0.05	0.025	0.01
1	1.32	2.71	3.84	5.02	6.64
2	2.77	4.61	5.99	7.38	9.21
3	4.11	6.25	7.81	9.35	11.34
4	5.39	7.78	9.49	11.14	13.28
5	6.63	9.24	11.07	12.83	15.09
6	7.84	10.64	12.59	14.45	16.81
7	9.04	12.02	14.07	16.01	18.48
8	10.22	13.36	15.51	17.53	20.09
9	11.39	14.68	16.92	19.02	21.67
10	12.55	15.99	18.31	20.48	23.21
11	13.70	17.28	19.68	21.92	24.72
12	14.85	18.55	21.03	23.34	26.22
13	15.98	19.81	22.36	24.74	27.69
14	17.12	21.06	23.68	26.12	29.14
15	18.25	22.31	25.00	27.49	30.58
16	19.37	23.54	26.30	28.85	32.00
17	20.49	24.77	27.59	30.19	33.41
18	21.60	25.99	28.87	31.53	34.81
19	22.72	27.20	30.14	32.85	36.19
20	23.83	28.41	31.41	34.17	37.57
22	26.04	30.81	33.92	36.78	40.29
24	28.24	33.20	36.42	39.36	42.98
26	30.43	35.56	38.89	41.92	45.64
28	32.62	37.92	41.34	44.46	48.28
30	34.80	40.26	43.77	46.98	50.89

# Solutions to exercises

- 3.1 The value of the mean is  $9.803 \text{ m s}^{-2}$ , and the values of the residuals in units of  $10^{-2} \text{ m s}^{-2}$  are 1, -1, 4, 1, -5, -1, 3, giving  $\sum d_i^2 = 54$ . Dividing this by  $n-1=6$ , and taking the square root, gives  $\sigma = 0.030 \text{ ms}^{-2}$ .  $\sigma_m = 0.030/\sqrt{7} = 0.011 \text{ ms}^{-2}$ . The group result is thus  $g = 9.803 \pm 0.011 \text{ m s}^{-2}$ .

If you have a calculator programmed to calculate standard deviations, try feeding in the values of  $g$ , and check that you get the same value for  $\sigma$ . Note that the calculator gives  $\sigma$ , and not  $\sigma_m$ . Some calculators are programmed to calculate  $s$  (equation 3.14), rather than  $\sigma$ , i.e. the quantity  $\sum d_i^2$  is divided by  $n$  instead of  $n-1$ . You can check your own calculator by feeding in the numbers 9 and 11, for which  $\sigma = \sqrt{2}$  and  $s = 1$ .

- 3.2 The values of  $E$  and the standard errors are given in units of  $10^{11} \text{ Nm}^{-2}$ . Newton's rings experiment. The mean value of  $E$  is 1.98. The value of  $\sigma$  is 0.25. Dial indicator experiment. The mean value of  $E$  is 2.047. The value of  $\sigma$  is 0.028.

Dividing  $\sigma$  by  $\sqrt{10}$  to obtain  $\sigma_m$  we have

Newton's rings experiment	$E = 1.98 \pm 0.08$
Dial indicator experiment	$E = 2.047 \pm 0.009$

The difference between the two mean values is slightly less than the standard error in the Newton's rings value, so there is little evidence for a systematic difference between the two experimental methods.

- 3.3 (a) 0.00266, (b) 0.0161, (c) 0.00036,  
(d) 0.683, (e) 0.954, (f) 0.997.

The fraction of readings between  $x$  and  $x + dx$  is  $f(z) dz$ , where  $f(z)$  is defined on p. 25 and tabulated in Appendix H.1;  $z = x/\sigma$ . For exercises (a) to (c)  $dz = 0.1/15.0$ . So the answer to (a) is  $0.399/150 = 0.00266$ . The answers to (d), (e), (f) are given by the values of  $\phi(z)$  for  $z = 1, 2, 3$ .

- 3.4 For the distribution of single measurements, the mean is

$$\mu = (-1 \times 0.9) + (9 \times 0.1) = 0.$$

Since the mean is zero, the variance  $\sigma^2$  is the mean square value of the measurements themselves, weighted by their probabilities, i.e.

$$\sigma^2 = [(-1)^2 \times 0.9] + [9^2 \times 0.1] = 9.$$

For a sample of three values we have to consider the possible combinations of -1



and +9, of which there are four. They are listed here with their probabilities and means.

Sample	Probability	Mean
-1, -1, -1	$(0.9)^3 = 0.729$	-1
-1, -1, 9	$(0.9)^2 \times (0.1) \times 3 = 0.243$	7/3
-1, 9, 9	$(0.9) \times (0.1)^2 \times 3 = 0.027$	17/3
9, 9, 9	$(0.1)^3 = 0.001$	9

For the distribution of the means of three values, the mean is

$$\mu_m = (-1 \times 0.729) + (7/3 \times 0.243) + (17/3 \times 0.027) + (9 \times 0.001) = 0,$$

as we would expect. The standard deviation  $\sigma_m$  is given by

$$\sigma_m^2 = [(-1)^2 \times 0.729] + [(7/3)^2 \times 0.243] + [(17/3)^2 \times 0.027] + [9^2 \times 0.001] = 3.$$

Thus  $\sigma_m = \sigma/\sqrt{3}$ .

**4.1** In the following solutions,  $g(A)$  means the % standard error in  $A$ , i.e.

$$g(A) = 100\Delta A/A.$$

$$\begin{aligned} \text{(a)} \quad g(A) &= 4, & g(A^2) &= 8, \\ Z &= 625, & \Delta Z &= 625 \times 8/100 = 50, \\ Z &= \underline{625 \pm 50}. \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad 2B &= 90 \pm 4, & \Delta Z &= (3^2 + 4^2)^{1/2} = 5, \\ Z &= \underline{10 \pm 5}. \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad g(C) &= 1, & g(C^2) &= 2, & C^2 &= 2500 \pm 50, \\ g(D) &= 8, & g(D^{\frac{1}{2}}) &= 12, & D^{\frac{1}{2}} &= 1000 \pm 120. \end{aligned}$$

$$\begin{aligned} \text{Put} \quad E &= C^2 + D^{\frac{1}{2}} = 3500, \\ \Delta E &= (50^2 + 120^2)^{1/2} = 130, & g(E) &= 3.7, \\ g(A) &= 3, & g(B) &= 5, \\ g(Z) &= (3^2 + 5^2 + 3.7^2)^{1/2} = 6.9, \\ Z &= \underline{350 \pm 24}. \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad \frac{\Delta(\ln B)}{\ln B} &= \frac{\Delta B/B}{\ln B} = \frac{0.02}{4.61} = \frac{0.43}{100}, \\ g(Z) &= (0.6^2 + 0.43^2)^{1/2} = 0.74, \\ Z &= \underline{46.1 \pm 0.3}. \end{aligned}$$

$$\begin{aligned} \text{(e)} \quad g(A) &= 4, & g\left(\frac{1}{A}\right) &= 4, \\ \frac{1}{A} &= 0.0200 \pm 0.0008, \\ Z &= \underline{0.9800 \pm 0.0008}. \end{aligned}$$

**4.2** (a) The measured values of  $l_x$ ,  $l_y$ ,  $l_z$  are independent. From Table 4.1 (ii), the standard error in the volume is therefore

$$\frac{\sqrt{3}}{100} \approx 0.02\%.$$

(b) The measured values are not independent. An increase in temperature causes

all three lengths to increase by the same fractional amount. The situation is represented by (4.9) with  $n = 3$ . The standard error in the volume is therefore 0.03%.

In practice, variations in the measured values of the length could be due to both instrumental errors and temperature fluctuations. It would be necessary to decide the contributions of each in order to estimate the error in the volume.

- 4.3 The values and standard errors of the slope are:

Method	Slope / $\mu\text{m kg}^{-1}$
least squares	$-349.2 \pm 1.9$
points in pairs	$-350.1 \pm 2.0$

The calculation by the method of least squares is made easier if the values of  $x$  are taken to be  $4W$ , where  $W$  is in kg. Then  $\bar{x}$  and all the  $(x_i - \bar{x})$  are simple integers.

- 4.4 The values and standard errors of the slope are

Method	Slope / $\text{mV K}^{-1}$
least squares	
errors only in $V$	$2.551 \pm 0.041$
errors only in $T$	$2.556 \pm 0.041$
points in pairs	$2.542 \pm 0.053$

The values are given to 3 decimal places for purposes of comparison, but normally these results would be quoted to only 2 decimal places.

Notice that the results of the two least-squares calculations are quite close. In fact there are probably errors in both the voltage and temperature readings, but a calculation that takes this into account requires a knowledge of the relative errors in  $V$  and  $T$ . The value of the slope of the best line when there are errors in both variables always lies between the two values corresponding to errors in only one or other of the variables (Guest 1961, p. 131). Since these two values are usually quite close, calculations are usually done on the assumption that only one of the variables is subject to error.

- 4.5 Weight each value of  $\tau$  inversely as the square of its standard error  $\Delta\tau$ , putting the weight  $w = 10/(\Delta\tau)^2$ .

$\tau/\text{s}$	$\Delta\tau/\text{s}$	$w$	$w\tau/\text{s}$
887.6	3.0	1.11	986
893.5	5.3	0.36	318
888.4	3.3	0.92	816
882.6	2.7	1.37	1211

The weighted mean is  $\bar{\tau} = \frac{\sum w\tau}{\sum w} = \frac{3331}{3.76} = 886.5 \text{ s}$ .

To calculate the standard error in  $\bar{\tau}$  we note that a weight of 1.11 corresponds to a standard error of 3.0 s. The weight of  $\bar{\tau}$  is 3.76, so its standard error is

$$\left[ \frac{1.11}{3.76} \right]^{\frac{1}{2}} \times 3.0 = 1.6 \text{ s}.$$

Thus  $\bar{\tau} = 886.5 \pm 1.6 \text{ s}$ .

4.6 From (4.30)  $d_i = y_i - mx_i - c$ .

Thus 
$$\sum d_i = \sum y_i - m \sum x_i - nc$$
  

$$= 0 \quad \text{from (4.26).}$$

Multiply the expression for  $d_i$  above by  $x_i$  and sum over  $i$ . This gives

$$\sum x_i d_i = \sum x_i y_i - m \sum x_i^2 - c \sum x_i$$
  

$$= 0 \quad \text{from (4.25).}$$

It may be noted that (4.26) comes from the relation  $\partial S / \partial c = 0$ . Thus the relation  $\sum d_i = 0$  follows from the presence of the constant  $c$  in the function  $y = mx + c$ . For the best line through the origin,  $y = mx$ , the relation  $\sum d_i = 0$  does not hold. The relation  $\sum x_i d_i = 0$  holds in both cases.

4.7 There is no error in  $x_i$ . Therefore the error in  $m_i = y_i/x_i$  is  $\Delta m_i = \Delta y_i/x_i$ . The error  $\Delta y_i$  is the same for all the points. Therefore  $\Delta m_i \propto 1/x_i$ , giving a weight  $w_i \propto x_i^2$ . The weighted mean of the  $y_i/x_i$  values is thus

$$\frac{\sum x_i^2 \frac{y_i}{x_i}}{\sum x_i^2} = \frac{\sum x_i y_i}{\sum x_i^2}.$$

5.1 (a) 
$$\rho = \frac{M}{abc}.$$

The fractional error in  $\rho$  due to each measured quantity is equal to the fractional error in the quantity. The fractional error in  $b = 10\%$  – entirely dominates the others. The error in  $\rho$  is therefore  $10\%$ . (The term 'error' in this and following solutions refers to standard error.)

(b) 
$$a^2 = 6400 \pm 160 \text{ mm}^2,$$
  

$$b^2 = 100 \pm 20 \text{ mm}^2.$$

The error in  $a^2$  now dominates that of  $b^2$ . The error in  $a^2 + b^2$  is  $2.5\%$ . The error in  $M$  is negligible by comparison. The error in  $I$  is therefore  $2.5\%$  and comes entirely from the error in  $a$ .

5.2 The error in  $\phi/C$  is  $3\%$ . The error in  $r$  is  $2\%$ ; so the error in  $r^4$  is  $8\%$ . The error in  $I$  is negligible. The error in  $n$  is  $(3^2 + 8^2)^{1/2} = 8.5\%$ . Therefore

$$n = (8.0 \pm 0.7) \times 10^{10} \text{ N m}^{-2}.$$

Notice that a quantity raised to a high power needs to be measured relatively precisely.

5.3 Substitution gives

$$A = \frac{T_1^2 + T_2^2}{H} = 8.0263 \text{ s}^2 \text{ m}^{-1},$$
  

$$B = \frac{T_1^2 - T_2^2}{h_1 - h_2} = 0.0199 \text{ s}^2 \text{ m}^{-1},$$
  

$$Z = \frac{8\pi^2}{g} = A + B = 8.0462 \text{ s}^2 \text{ m}^{-1}.$$

The calculation of the error is straightforward. An interesting feature is that though  $A$  is very much larger than  $B$ , the error due to  $T_1$  comes largely from the  $B$  term. The latter contributes  $(h_1 + h_2)/2h_1 = \frac{5}{7}$  of the error due to  $T_1$ . The errors from  $h_1$  and  $h_2$  are negligible. The errors in  $Z$  due to the errors in  $T_1, T_2, H$  are 7, 3,  $3 \times 10^{-4} \text{ s}^2 \text{ m}^{-1}$  and these combine to give a total error of  $8 \times 10^{-4} \text{ s}^2 \text{ m}^{-1}$ . The final result is

$$g = 9.8129 \pm 0.0010 \text{ m s}^{-2}.$$

- 5.4 Calculate  $\mu$  for the three pairs of values (i)  $A, D$ ; (ii)  $A + \Delta A, D$ ; (iii)  $A, D + \Delta D$ . From the sine values of  $(A + D)/2$  and  $A/2$ , the three values of  $\mu$  are

$$\frac{0.7435}{0.5023}, \quad \frac{0.7445}{0.5035}, \quad \frac{0.7455}{0.5023}.$$

These give  $\Delta\mu_A = 0.002$ ,  $\Delta\mu_D = 0.004$ ,  
whence  $\mu = 1.480 \pm 0.004$ .

- 5.5  $\mu x = \ln I_0 - \ln I = 0.7829$ .

The simplest way of finding the error is that of direct substitution. Thus

$$\begin{aligned} \ln(I_0 + \Delta I_0) - \ln I_0 &= 0.006, \\ \ln(I + \Delta I) - \ln I &= 0.011. \end{aligned}$$

These give a combined error of 0.012, i.e.

$$\mu x = 0.783 \pm 0.012.$$

$x = 10 \text{ mm}$  with negligible error. Therefore

$$\mu = 78.3 \pm 1.2 \text{ m}^{-1}.$$

The expression for the error given by the formal method (section 4.1 (b)) is

$$(\Delta\mu)^2 = \frac{1}{x^2} \left[ \left( \frac{\Delta I}{I} \right)^2 + \left( \frac{\Delta I_0}{I_0} \right)^2 \right] + \left( \frac{\mu \Delta x}{x} \right)^2,$$

which is not difficult to evaluate. (Again the term in  $\Delta x$  is negligible.) But you have to be able to derive the expression correctly in the first place. You are less liable to make mistakes with the simpler method of direct substitution.

- 5.6 As in exercise 5.5 the error can be evaluated either by direct substitution or formally. Since  $n$  and  $d$  are fixed,  $\lambda \propto \sin \theta$ . Evaluate  $\sin \theta$  for  $\theta$  and  $\theta + \Delta\theta$ . The results are 0.1959 and 0.1985, giving  $\Delta\lambda/\lambda = 1.3\%$ . Since  $E \propto (\text{momentum})^2 \propto 1/\lambda^2$ ,  $\Delta E/E = 2\Delta\lambda/\lambda = \underline{2.6\%}$ . The expression given by the formal method is  $\Delta E/E = 2 \cot \theta \Delta\theta$ , where

$$\Delta\theta = \frac{9}{60} \times \frac{\pi}{180} \text{ rad}.$$

- 5.7 We have

$$L \propto \frac{f^2}{E}.$$

Denote the fractional increases in  $f$  and  $E$  by  $r_f$  and  $r_E$ . Then the fractional increase in  $L$  is

$$r_L = 2r_f - r_E.$$

Since the rise in temperature is 10 K,

$$\begin{aligned}\alpha &= r_L/10 \\ &= (-0.500 + 0.520) \times 10^{-2} \times 10^{-1} \\ &= \underline{20 \times 10^{-6}}.\end{aligned}$$

The error in  $2r_f$  is  $4 \times 10^{-5}$ ; the error in  $r_E$  is  $3 \times 10^{-5}$ . The error in  $r_L$  is therefore  $5 \times 10^{-5}$  and the error in  $\alpha$  is  $\underline{5 \times 10^{-6}}$ .

The method is a bad one. The two primary quantities  $r_f$  and  $r_E$  are measured to better than 1 part in 100, but the value of  $\alpha$  is only good to 1 part in 4. This is an example of Case II, p. 48.

6.1 (a) Let the interval between flashes be

$$T_0 = \frac{1}{f_0}.$$

Suppose that  $f$  is slightly greater than  $mf_0$ . Then in time  $T_0$  the object will make  $m$  revolutions plus a small fraction  $\delta$  of a revolution. The body is actually rotating with frequency

$$f = \frac{m + \delta}{T_0} = (m + \delta)f_0,$$

but it appears to be rotating with frequency

$$f_{\text{app}} = \frac{\delta}{T_0} = \delta f_0 = f - mf_0.$$

The sense of the apparent rotation is the same as that of the actual rotation. If  $f$  is slightly less than  $mf_0$ , the quantity  $\delta$  is negative and therefore  $f_{\text{app}}$  is negative. The body appears to be rotating in the opposite sense to that of the actual rotation.

If  $f$  is exactly equal to  $mf_0$  the body appears stationary. This can sometimes happen with fluorescent lighting, where the variation of illumination as the mains voltage varies sinusoidally is much more marked than with a filament bulb. For this reason it is dangerous to have fluorescent lighting in workshops with machinery rotating at high speeds.

$$\begin{aligned}\text{(b)} \quad f &= mf_0 + f_{\text{app}}, \\ mf_0 &= 500.00 \pm 0.05 \text{ Hz}, \\ f_{\text{app}} &= 0.40 \pm 0.05 \text{ Hz}.\end{aligned}$$

Therefore

$$f = 500.40 \pm 0.07 \text{ Hz}.$$

*Comments*

The stroboscope, like the beat method of measuring frequency, is an example of the Case 1 situation where we measure the difference between two nearly equal quantities, one of which is known to high precision. Another example is the *doublet method* in mass spectroscopy, where the mass difference between two ions of nearly equal mass, such as the deuterium atom and the molecule of light hydrogen, is measured, and the value combined with the precisely known mass of one of them. See Squires 1998 for further discussion of the method.

6.2 For 20 swings,  $t = 40.8 \pm 0.2$  s. Therefore

$$T = 2.04 \pm 0.01 \text{ s.} \quad (1)$$

With this approximate value of  $T$ , we see that  $t = 162.9$  s corresponds to about  $N_1 = 80$  swings. Try  $N_1 = 79, 80, 81$ . The results are

$$N_1 = 79, \quad T = 2.0620 \pm 0.0025 \text{ s,}$$

$$N_1 = 80, \quad T = 2.0363 \pm 0.0025 \text{ s,}$$

$$N_1 = 81, \quad T = 2.0111 \pm 0.0025 \text{ s.}$$

Only the value

$$T = 2.0363 \pm 0.0025 \text{ s} \quad (2)$$

is consistent with (1). So  $N_1 = 80$ , and we now have a more precise value of  $T$ , which serves to fix the value of  $N$  for the next measurement.

The time  $t = 653.6$  s corresponds to about 320 swings. We try values of  $N_2$  in this region. The results are

$$N_2 = 319, \quad T = 2.0489 \pm 0.0006 \text{ s,}$$

$$N_2 = 320, \quad T = 2.0425 \pm 0.0006 \text{ s,}$$

$$N_2 = 321, \quad T = 2.0361 \pm 0.0006 \text{ s,}$$

$$N_2 = 322, \quad T = 2.0298 \pm 0.0006 \text{ s.}$$

Only the value

$$T = 2.0361 \pm 0.0006 \text{ s} \quad (3)$$

is consistent with (2). So  $N_2 = 321$ , and the value of  $T$  is given by (3).

*Comments*

Clearly, the principle of the method is that each measurement gives a value of  $T$ , which enables us to determine the integer for the following measurement, and thus obtain a more precise value of  $T$ . Only the swings in the initial measurement need be counted. Note that the factor by which the timing period is increased in successive measurements – 4 in the present exercise – depends on the value of  $\Delta t$ . If this error were larger, the factor would have to be reduced in order to obtain an unambiguous value of the integer.

The method of *exact fractions* in high-precision interferometry is based on similar reasoning. In this method the optical path length in an interferometer is

determined for monochromatic light of three known wavelengths  $\lambda_1, \lambda_2, \lambda_3$ . The path length is given by

$$d = (n_1 + f_1)\lambda_1 = (n_2 + f_2)\lambda_2 = (n_3 + f_3)\lambda_3, \quad (4)$$

where the three  $n$ s are integers and the three  $f$ s are fractions. Only the three fractions are actually measured, but an approximate knowledge of  $d$  suffices to identify the three integers, and hence to determine  $d$  to much higher precision. See Ditchburn 1952, p. 283 for details of the method.

- 7.1 For a sinusoidal voltage of angular frequency  $\omega = 2\pi f$ , the impedance of the capacitor is  $1/j\omega C$ .  $j = \sqrt{-1}$ . Therefore

$$\frac{V_C}{V_Q} = \frac{1/j\omega C}{(1/j\omega C) + R} = \frac{1}{1 + j\omega CR}$$

$$\left| \frac{V_C}{V_Q} \right|^2 = \frac{1}{(1 + j\omega CR)(1 - j\omega CR)} = \frac{1}{1 + \omega^2 C^2 R^2},$$

which gives the required result.

- 7.2  $\Delta V_z = (dV_z/df_z)\Delta f_z$   
 $= 3 \times 0.02 \times 10^{-3} = 6 \times 10^{-5} \text{ V}.$

Therefore  $\Delta V_z/V_z = 10^{-5}$ .

- 7.3 Outside the Earth's surface,  $g$  is proportional to  $1/R^2$ , where  $R$  is the distance from the centre of the Earth. Therefore a decrease in  $g$  of  $\Delta g$  corresponds to an increase in height  $\Delta h$ , given by

$$\frac{\Delta g}{g} = 2 \frac{\Delta h}{R_E},$$

where  $R_E$ , the radius of the Earth, is 6400 km. Thus

$$\Delta h = \frac{1}{2} \times 6.4 \times 10^6 \times 10^{-8} = 32 \text{ mm}.$$

- 12.1 The values used for the atomic constants in these solutions are approximate ones that you will find it useful to remember. In fact the values of these constants are probably known to a few parts in  $10^7$ .

(a) 60 W.

The thermal conductivity of copper at  $0^\circ\text{C}$  is  $385 \text{ W m}^{-1} \text{ K}^{-1}$ . The calculation assumes no heat loss along the rod.

(b) The readings are too low by about 0.010%. The linear expansivity is about  $10$  to  $11 \times 10^{-6}$  for most kinds of steel.

(c) (i)  $0.0199 \Omega$ . (ii) It increases by  $0.0017 \Omega$ .

The resistivity of copper is  $1.56 \times 10^{-8} \Omega \text{ m}$  at  $0^\circ\text{C}$  and it increases by 0.4% per degree rise in temperature.

(d) 4.1 mV.

(e) (i)  $0.25 \text{ m s}^{-1}$ . (ii)  $0.45 \text{ m s}^{-1}$ .

At  $20^\circ\text{C}$  the viscosity of water is  $1.00 \times 10^{-3} \text{ N s m}^{-2}$ , and at  $50^\circ\text{C}$  it is  $0.55 \times 10^{-3} \text{ N s m}^{-2}$ .

(f)  $13 \text{ kN}$ .

The Young modulus of steel is  $2.1 \times 10^{11} \text{ N m}^{-2}$ .

(g)  $1.29 \text{ m}$ .

The speed of sound in air at  $0^\circ\text{C}$  is  $331 \text{ m s}^{-1}$ .

(h)  $1.9 \text{ km s}^{-1}$ .

The result may be obtained from the equation

$$\frac{1}{2}mv^2 = \frac{3}{2}kT.$$

$m$ , the mass of the hydrogen molecule, is  $2 \times 1.67 \times 10^{-27} \text{ kg}$ , and  $k$ , the Boltzmann constant, is  $1.38 \times 10^{-23} \text{ J K}^{-1}$ .

(i)  $G = 6.7 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$ .

Use the equation

$$g = \frac{GM}{R_E^2},$$

where  $M$  is the mass and  $R_E$  the radius of the Earth. The mean density of the Earth is  $5500 \text{ kg m}^{-3}$ , and its radius is  $6400 \text{ km}$ .  $g = 9.81 \text{ m s}^{-2}$ .

(j) Typical wavelengths for red, green, and violet light are  $700 \text{ nm}$ ,  $550 \text{ nm}$ , and  $400 \text{ nm}$ . The values give (i) 710 lines per mm, (ii)  $23.1^\circ$ , (iii)  $16.6^\circ$ .

(k)  $18 \text{ W}$ .

Stefan-Boltzmann constant  $\sigma = 5.67 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$ .

(l) (i)  $1.9 \times 10^7 \text{ m s}^{-1}$ . (ii)  $39 \text{ pm}$ .

mass of electron  $m_e = 9.1 \times 10^{-31} \text{ kg}$ ,

elementary charge  $e = 1.6 \times 10^{-19} \text{ C}$ ,

Planck constant  $h = 6.6 \times 10^{-34} \text{ J s}$ .

(m)  $0.29 \text{ T}$ .

mass of proton  $m_p = 1.67 \times 10^{-27} \text{ kg}$ .

(n)  $91 \text{ nm}$ .

The answer is the reciprocal of the Rydberg constant

$$R = 1.10 \times 10^7 \text{ m}^{-1}.$$

(o)  $931 \text{ MeV}$ .

The mass in kilograms of a particle of atomic weight unity is  $10^{-3}/N_A$ , where  $N_A = 6.0 \times 10^{23}$  is the Avogadro constant.

speed of light  $c = 3.0 \times 10^8 \text{ m s}^{-1}$ .

## 12.2

(a)  $(1 + \delta_1)(1 + \delta_2)(1 + \delta_3) \approx 1 + \delta_1 + \delta_2 + \delta_3$ , ---

if the  $\delta$ s are small compared with 1. So the answer is

$$1 + 0.00025 + 0.00041 - 0.00013 = \underline{1.00053}.$$

(b)  $\frac{1}{(1 + \delta)^2} \approx 1 - 2\delta$  if  $\delta \ll 1$ .



$\delta$  is approximately 9/72 000, so 912.64 must be reduced by 18 parts in 72 000, that is, by 1 part in 4000, which is 0.23. The answer is 912.41.

$$\begin{aligned} \text{(c)} \quad (9.100)^{\frac{1}{2}} &= 3 \times \left(1 + \frac{1}{90}\right)^{\frac{1}{2}} \\ &\approx 3 \times \left(1 + \frac{1}{180}\right) \\ &= \underline{3.017}. \end{aligned}$$

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